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Brief paper

Input-to-state stability and inverse optimality of predictor feedback for multi-input linear systems[☆]

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ABSTRACT

For the "exact" predictor-feedback control design, recently introduced by Tsubakino, Krstic, and Oliveira for multi-input linear systems with distinct input delays, we establish input-to-state stability, with respect to additive plant disturbances, as well as robustness to constant multiplicative uncertainties affecting the inputs. We also show that the exact predictor-feedback controller is inverse optimal with respect to a meaningful differential game problem. Our proofs capitalize on the availability of a backstepping transformation, which is formulated appropriately in a recursive manner. An example, computed numerically, is provided to illustrate the validity of the developed results.

1. Introduction

Although for multi-input linear systems with distinct input delays predictor-based control designs have been developed since the late 1970s and early 1980s, (see, for example, Artstein, 1982; Manitius & Olbrot, 1979; Tsubakino, Krstic, & Oliveira, 2016). It was not until the result in Tsubakino et al. (2016) that an "exact" predictor-feedback control design has appeared. This predictor-feedback controller is referred to as exact, to highlight the fact that each of the control input signals employs, in the nominal (for the delay-free system) feedback law, the predictor of the state as many time units in the future as the corresponding input delay. This key idea has enabled the development of extensions to nonlinear systems (Bekiaris-Liberis & Krstic, 2017), to systems with simultaneous input and state delays (Bresch-Pietri & Di Meglio, 2017; Kharitonov, 2017), and to extremum seeking control for static maps with delays (Oliveira, Krstic, & Tsubakino, 2017).

In the single-delay linear case, the inverse optimality and disturbance attenuation properties of the basic predictor feedback as well as its low-pass-filtered modification are studied in Cai, Bekiaris-Liberis, and Krstic (2018) and Krstic (2008), whereas for nonlinear systems respective developments can be found, for instance, in Cai, Lin, and Liu (2015) and Karafyllis and Krstic (2017). Robustness of predictor feedback to delay mismatches, for both linear and nonlinear systems with a single input delay, is studied in Bekiaris-Liberis and Krstic (2013), Karafyllis and Krstic (2013) and Krstic (2008). When uncertainties in the plant parameters or the delay are large, adaptive prediction-based schemes may be employed, which are recently developed for systems with a single (Basturk & Krstic, 2015; Bresch-Pietri, Chauvin, & Petit, 2012; Bresch-Pietri & Krstic, 2014; Zhu, Krstic, & Su, 2017) or multiple (Zhu, Krstic, & Su, 2018) delays. Prediction-based control designs for single-delay systems under sampling also exist (Karafyllis & Krstic, 2013; Mazenc & Normand-Cyrot, 2013).

Besides highlighting some of the benefits of the exact predictor-feedback scheme and the accompanying backstepping transformation, the problem we tackle in the present paper is inspired by highway traffic control problems. In particular, in scenarios where the goal is to regulate the flow (ODE state) at a potential bottleneck area, far downstream from the locations of actuated on-ramps whose flows (control inputs) may be manipulated (via, for example, ramp metering) and where the mainstream inflow (plant disturbance) to the highway is unmeasured, see, for instance (Wang, Kosmatopoulos, Papageorgiou, & Papamichail, 2014). Other applications in which multi-input systems with several delays may appear include network congestion control (Quet et al., 2002; Tregouet, Seuret, & Di Loreto,

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2016), robotic manipulators (Ailon, 2004), multi-agent systems (Abdessameud & Tayebi, 2011) and autonomous ground vehicles (Malisoff & Zhang, 2013), to name only a few (Donkers, Daafouz, & Heemels, 2014; Fridman, 2014; Mahjoub, Van Assche, Giri, & Chaoui, 2015).

Motivated by these specific applications, for the exact predictor-feedback controller in the present work we establish (1) input-to-state stability with respect to additive plant disturbances, (2) robustness to constant multiplicative uncertainties affecting the inputs, and (3) inverse optimality with respect to a meaningful differential game problem. All of these results for multi-input linear systems with distinct input delays under predictor feedback are novel. Our proofs are based on a recursive formulation of the infinite-dimensional backstepping transformation and the construction of a Lyapunov functional. A simulation example of an unstable third-order system with two delays is also provided to illustrate the validity of the presented analysis.

Notation. For an n-vector, $|\cdot|$ denotes the Euclidean norm. For a matrix $A = (a_{ij})_{n\times m}$, $|A|$ denotes the induced matrix norm. For functions $u_i : [0, D_i] \times \mathbb{R} \to \mathbb{R}$ and $U_i : \mathbb{R} \to \mathbb{R}, i = 1, \ldots, m$, we denote $\|u_i(t)\| = \left(\int_0^{D_i} u_i(x,t)^2 dx\right)^{1/2}$ and $\|U_i(t)\| = \left(\int_{t-D_i}^{t} U_i(\theta)^2 d\theta\right)^{1/2}$, respectively.

2. System description and control law design

Consider the following system:

$$\dot{X}(t) = AX(t) + \sum_{i=1}^{m} b_i U_i(t - D_i) + B\delta(t), \tag{1}$$

where $X \in \mathbb{R}^n$ is the state, $U_1, \ldots, U_m \in \mathbb{R}$ are control inputs, $D_1, \ldots, D_m$ are input delays satisfying (without loss of generality) $0 < D_1 \le \cdots \le D_m$, A is an $n \times n$ matrix, $b_i, i = 1, \ldots, m$ are n-dimensional vectors, B is an $n\times l$ matrix, and $\delta \in \mathbb{R}^l$ is disturbance. We assume that the pair $([A, b_1, \ldots, b_m])$ is stabilizable. In the delay-free case of system (1), we choose the following linear feedback control law:

$$\overline{U}_i(t) = k_i^T X(t), \tag{2}$$

where each vector $k_i \in \mathbb{R}^n, i = 1, 2, \ldots, m$, is selected so that $A + \sum_{i=1}^{m} b_i k_i^T$ is Hurwitz.

We consider the following basic predictor-feedback control law:

$$U_i(t) = \frac{c_i}{c_i + 1}\overline{U}_i(t) = U_i^*(t), \tag{3}$$

where $c_i > 0,\ i = 1, 2, \ldots, m$, are sufficiently large constants and $\overline{U}_i(t)$ are given in Tsubakino et al. (2016) as

$$\overline{U}_i(t) = k_i^T P_i(t),\ \ i = 1, 2, \ldots, m, \tag{4}$$

where the predictors are given by

$$P_1(t) = e^{AD_1}X(t) + \int_{t-D_1}^{t} e^{A(t-s)} \sum_{i=1}^{m} b_i U_i(s - D_{i,1})ds, \tag{5}$$

$$P_2(t) = e^{A_1 D_{2,1}}P_1(t) + \int_{t-D_{2,1}}^{t} e^{A_1(t-s)} \sum_{i=2}^{m} b_i U_i(s - D_{i,2})ds, \tag{6}$$

$$\vdots$$

$$P_m(t) = e^{A_{m-1}D_{m,m-1}}P_{m-1}(t) + \int_{t-D_{m,m-1}}^{t} e^{A_{m-1}(t-s)} b_m U_m(s)ds, \tag{7}$$

the matrices $A_i, i = 1, \ldots, m$, are

$$A_i = A + \sum_{j=1}^{i} b_j k_j^T, \tag{8}$$

and $D_{j,i} = D_j - D_i$, for all $i \le j \le m$, with $D_0 = 0$.

3. Gain-robustness and inverse optimality of the basic predictor feedback controller

We first prove that the closed-loop system (1), (3)–(7) is input-to-state stable (ISS) and we then show the inverse optimality of (3)–(7), when the c_i's are sufficiently large.[1]

3.1. ISS of the basic predictor-feedback controller

Theorem 1. *Consider the closed-loop system consisting of (1) with the control laws (3)–(7). There exists $c^* > 0$ such that the closed-loop system is ISS provided that $\underline{c} = \min_{i=1,2,\ldots,m} c_i > c^*$, that is, there exist positive constants $\psi, \overline{\lambda}$, and $\zeta > 0$, such that for all $\underline{c} > c^*$,*

$$\Omega(t) \le \psi\Omega(0)e^{-\overline{\lambda} t} + \zeta\left(\sup_{0\le\tau\le t} |\delta(\tau)|\right)^2, \quad \text{for all } t \ge 0, \tag{9}$$

with

$$\Omega(t) = |X(t)|^2 + \sum_{i=1}^{m} \|U_i(t)\|^2. \tag{10}$$

Remark 1. Theorem 1 shows that the basic predictor-feedback controller (3)–(7), besides being input-to-state stabilizing with respect to additive plant disturbances, is robust to constant multiplicative uncertainty affecting the system's inputs. Moreover, if the control law (3) is modified to

$$U_i(t) = \frac{c_i + 1}{c_i}\overline{U}_i(t), \quad i = 1, 2, \ldots, m, \tag{11}$$

then the result of Theorem 1 still holds. In other words, the basic predictor-feedback controller is robust to uncertainties that are both larger and smaller than unity. Since such a result could be established employing identical arguments to the proof of Theorem 1, its proof is omitted as the superfluous technical details would only distract the reader from the substance of the result, which is robustness of predictor feedback.

Remark 2. When the control gains $k_i\frac{c_i}{c_i+1}$ in (3) are replaced by $k_i + \triangle_i$ where $|\triangle_i|, i = 1, 2, \ldots, m$, are sufficiently small, the result of Theorem 1 still holds. The proof of such a result would be almost identical to that of Theorem 1.

Remark 3. The closed-loop system in Tsubakino et al. (2016) is not the same with the closed-loop system (1) under (3)–(7), with $\delta \equiv 0$, and thus, the result in Theorem 1 cannot follow combining the exponential stability result in Tsubakino et al. (2016) with the results in, for example, Dashkovskiy and Mironchenko (2013). It should be also noted that another advantage of performing the stability analysis adopting the constructive strategy of the proof of Theorem 1 is that one obtains explicit input-to-state stability

[1] Considering the system of retarded functional differential equations derived by differentiating (3)–(7) and assuming that the initial conditions $U_i(s)$, $-D_i \le s \le 0$, $i = 1, \ldots, m$, are absolutely continuous and compatible with the feedback laws (3)–(7), existence and uniqueness of an absolutely continuous solution $(X(t), U_1(t), \ldots, U_m(t))$, $t \ge 0$, $i = 1, \ldots, m$ to the closed-loop system (1), (3)–(7), may follow, e.g., from Theorem 5.2 in Kolmanovskii and Myshkis (1999) (for a measurable and bounded disturbance δ).

estimates, as estimate (9) with the specific constants ψ, $\bar{\lambda}$, and ζ, which is a result of the explicit construction of a Lyapunov functional.

The proof of Theorem 1 is based on a series of technical lemmas, which are presented next, together with transport PDE representation for the actuator state, which allows us to re-write system (1) as

$$\dot{X}(t) = AX(t) + \sum_{i=1}^{m} b_i u_i(0,t) + B\delta(t) \tag{12}$$

$$\partial_t u_i(x,t) = \partial_x u_i(x,t),\ x \in (0, D_i), \quad i = 1,2,\ldots,m \tag{13}$$

$$u_i(D_i,t) = U_i(t), \quad i = 1,2,\ldots,m, \tag{14}$$

where

$$u_i(x,t) = U_i(x+t-D_i), \quad i = 1,2,\ldots,m. \tag{15}$$

In this notation, we define

$$p_1(x,t) = e^{Ax}X(t) + \int_0^x e^{A(x-\alpha)} \sum_{i=1}^{m} b_i u_i(\alpha,t)d\alpha,\ \ 0 \le x \le D_1, \tag{16}$$

$$p_2(x,t) = e^{A_1(x-D_1)}p_1(D_1,t) + \int_{D_1}^x e^{A_1(x-\alpha)} \sum_{i=2}^{m} b_i u_i(\alpha,t)d\alpha,\ \ D_1 \le x \le D_2, \tag{17}$$

$$\vdots$$

$$p_m(x,t) = e^{A_{m-1}(x-D_{m-1})}p_{m-1}(D_{m-1},t) + \int_{D_{m-1}}^x e^{A_{m-1}(x-\alpha)} \times b_m u_m(\alpha,t)d\alpha, \quad D_{m-1} \le x \le D_m, \tag{18}$$

and thus, with this representation, (4) becomes

$$\overline{U}_i(t) = k_i^T p_i(D_i,t), \quad i = 1,2,\ldots,m. \tag{19}$$

From (16)–(18), it is also easy to see that

$$p_1(0,t) = X(t), \tag{20}$$

$$p_2(D_1,t) = p_1(D_1,t), \tag{21}$$

$$\vdots$$

$$p_m(D_{m-1},t) = p_{m-1}(D_{m-1},t). \tag{22}$$

Lemma 1. *The backstepping transformations of $u_i(x,t), i = 1,\ldots,m$, defined as*

$$\omega_1(x,t) = u_1(x,t) - k_1^T p_1(x,t),\ x \in [0,D_1] \tag{23}$$

$$\omega_2(x,t) = u_2(x,t) - \begin{cases} k_2^T p_1(x,t), & x \in [0,D_1] \\ k_2^T p_2(x,t), & x \in [D_1,D_2] \end{cases} \tag{24}$$

$$\vdots$$

$$\omega_m(x,t) = u_m(x,t) - \begin{cases} k_m^T p_1(x,t), & x \in [0,D_1] \\ k_m^T p_2(x,t), & x \in [D_1,D_2] \\ \vdots & \\ k_m^T p_m(x,t), & x \in [D_{m-1},D_m], \end{cases} \tag{25}$$

where $p_i(x,t), i = 1,2,\ldots,m$, are given by (16)–(18), together with the control laws (3), (19), (16)–(18) transform system (12)–(14) to the following "target system":

$$\dot{X}(t) = \left(A + \sum_{i=1}^{m} b_i k_i^T\right)X(t) + \sum_{i=1}^{m} b_i \omega_i(0,t) + B\delta(t) \tag{26}$$

$$\partial_t \omega_1(x,t) = \partial_x \omega_1(x,t) - k_1^T e^{Ax}B\delta(t), \quad x \in (0,D_1) \tag{27}$$

$$\partial_t \omega_2(x,t) = \partial_x \omega_2(x,t) - \begin{cases} k_2^T e^{Ax}B\delta(t), \quad x \in (0,D_1), \\ k_2^T e^{A_1(x-D_1)}e^{AD_1}B\delta(t) \\ -\frac{k_2^T}{c_1+1}e^{A_1(x-D_1)}b_1 k_1^T p_1(D_1,t), \\ \quad x \in (D_1,D_2) \end{cases} \tag{28}$$

$$\vdots$$

$$\partial_t \omega_m(x,t) = \partial_x \omega_m(x,t) - \begin{cases} k_m^T e^{Ax}B\delta(t), \quad x \in (0,D_1), \\ k_m^T e^{A_1(x-D_1)}e^{AD_1}B\delta(t) \\ -\frac{k_m^T}{c_1+1}e^{A_1(x-D_1)}b_1 k_1^T p_1(D_1,t),\ x \in (D_1,D_2) \\ \vdots \\ k_m^T e^{A_{m-1}(x-D_{m-1})}e^{A_{m-2}D_{m-1,m-2}} \\ \times e^{A_{m-3}D_{m-2,m-3}}\ldots e^{A_1 D_{2,1}}e^{AD_1}B\delta(t) \\ -\sum_{j=1}^{m-2}\Big\{k_m^T e^{A_{m-1}(x-D_{m-1})}e^{A_{m-2}D_{m-1,m-2}} \\ \times e^{A_{m-3}D_{m-2,m-3}}\ldots e^{A_j D_{j+1,j}}\frac{b_j k_j^T}{c_j+1}p_j(D_j,t)\Big\} \\ -k_m^T e^{A_{m-1}(x-D_{m-1})}\frac{b_{m-1}k_{m-1}^T}{c_{m-1}+1}p_{m-1}(D_{m-1},t), \\ \quad x \in (D_{m-1},D_m) \end{cases} \tag{29}$$

$$\omega_i(D_i,t) = -\frac{1}{c_i+1}k_i^T p_i(D_i,t), \quad i = 1,2,\ldots,m. \tag{30}$$

Proof. The space is limited, the proof is omitted.

Lemma 2. *The inverse backstepping transformations of (23)–(25) are defined by*

$$u_1(x,t) = \omega_1(x,t) + k_1^T q_1(x,t), \quad x \in [0,D_1] \tag{31}$$

$$u_2(x,t) = \omega_2(x,t) + \begin{cases} k_2^T q_1(x,t), & x \in [0,D_1] \\ k_2^T q_2(x,t), & x \in [D_1,D_2] \end{cases} \tag{32}$$

$$\vdots$$

$$u_m(x,t) = \omega_m(x,t) + \begin{cases} k_m^T q_1(x,t), & x \in [0,D_1] \\ k_m^T q_2(x,t), & x \in [D_1,D_2] \\ \vdots & \\ k_m^T q_m(x,t), & x \in [D_{m-1},D_m], \end{cases} \tag{33}$$

where

$$q_1(x,t) = e^{A_m x}X(t) + \int_0^x e^{A_m(x-\alpha)} \times \sum_{i=1}^{m} b_i \omega_i(\alpha,t)d\alpha, \quad 0 \le x \le D_1, \tag{34}$$

$$q_2(x,t) = e^{A_m(x-D_1)}q_1(D_1,t) + \int_{D_1}^x e^{A_m(x-\alpha)} \times \sum_{i=2}^{m} b_i \omega_i(\alpha,t)d\alpha, \quad D_1 \le x \le D_2, \tag{35}$$

$$\vdots$$

$$q_m(x,t) = e^{A_m(x-D_{m-1})}q_{m-1}(D_{m-1},t) + \int_{D_{m-1}}^x e^{A_m(x-\alpha)} \times b_m \omega_m(\alpha,t)d\alpha, \quad D_{m-1} \le x \le D_m. \tag{36}$$

Proof. It can be deduced using similar arguments to the corresponding proof in Tsubakino et al. (2016) (Appendix B).

Lemma 3. *There exist positive scalars γ_j and ι_j (independent of the c_j's), $j = 1, 2, \ldots, m$, such that*

$$\sup_{x\in[D_{j-1},\,D_j]} \left|p_j(x,t)\right|^2 \le \gamma_j \left(|X(t)|^2 + \sum_{i=1}^{m} \|u_i(t)\|^2\right), \tag{37}$$

$$\sup_{x\in[D_{j-1},\,D_j]} \left|q_j(x,t)\right|^2 \le \iota_j \left(|X(t)|^2 + \sum_{i=1}^{m} \|\omega_i(t)\|^2\right), \tag{38}$$

for all $j = 1, 2, \ldots, m$.

Proof. Noting that $0 < D_1 \le \cdots \le D_m$ and using Cauchy–Schwarz inequality, from (16)–(18) and (34)–(36), we can derive (37) and (38), respectively, with

$$\begin{aligned}\gamma_j &= 2^{j+1} e^{2|A_{j-1}|D_j} \cdots e^{2|A|D_1} \\ &\quad\times \max\left\{1, (m-j+1)D_{j,j-1} \max_{i=j,\ldots,m}\{|b_i|^2\}\right\}\cdots \\ &\quad\times \max\left\{1, mD_1 \max_{i=1,\ldots,m}\{|b_i|^2\}\right\},\end{aligned} \tag{39}$$

with $j = 1, 2, \ldots, m$, $A_0 = A$, and

$$\begin{aligned}\iota_j &= 2^{j+1} e^{2|A_m|(D_1+D_2+\cdots+D_j)} \\ &\quad\times \max\left\{1, (m-j+1)D_{j,j-1} \max_{i=j,\ldots,m}\{|b_i|^2\}\right\}\cdots \\ &\quad\times \max\{1, mD_1 \max_{i=1,\ldots,m}\{|b_i|^2\}\}, \quad j = 1, 2, \ldots, m.\end{aligned} \tag{40}$$

Lemma 4. *There exist positive constants α_1 and α_2 (independent of the c_i's) such that*

$$|X(t)|^2 + \sum_{i=1}^{m} \|\omega_i(t)\|^2 \le \alpha_1 \left(|X(t)|^2 + \sum_{i=1}^{m} \|u_i(t)\|^2\right), \tag{41}$$

$$|X(t)|^2 + \sum_{i=1}^{m} \|u_i(t)\|^2 \le \alpha_2 \left(|X(t)|^2 + \sum_{i=1}^{m} \|\omega_i(t)\|^2\right). \tag{42}$$

Proof. With Lemma 3 and relations (23)–(25), (31)–(33), we get (41), (42) with $\alpha_1 = 2\left(1 + \sum_{j=1}^{m} D_j|k_j|^2\gamma_j\right)$ and $\alpha_2 = 2\left(1 + \sum_{j=1}^{m} D_j|k_j|^2\iota_j\right)$, respectively.

Proof of Theorem 1. Since $A + \sum_{i=1}^{m} b_i k_i^T$ is Hurwitz, for any positive definite matrix S, there exists a unique positive definite matrix M such that

$$M\left(A + \sum_{i=1}^{m} b_i k_i^T\right) + \left(A + \sum_{i=1}^{m} b_i k_i^T\right)^T M = -S. \tag{43}$$

Consider a Lyapunov functional

$$V(t) = X(t)^T M X(t) + \frac{a_1}{2}\sum_{i=1}^{m}\int_0^{D_i} e^x \omega_i(x,t)^2 dx, \tag{44}$$

where the constant $a_1 > 0$ is determined later on. The derivative of $V(t)$ along the solutions of system (26)–(30) satisfies the following equality:

$$\begin{aligned}\dot V(t) &= -X^T(t)SX(t) + 2X^T(t)M\sum_{i=1}^{m} b_i\omega_i(0,t) \\ &\quad + 2X^T(t)MB\delta(t) \\ &\quad + a_1\sum_{i=1}^{m}\int_0^{D_i} e^x\omega_i(x,t)\partial_t\omega_i(x,t)dx.\end{aligned} \tag{45}$$

With (26)–(30), we compute the following integral for each i:

$$\begin{aligned}&\int_0^{D_i} e^x\omega_i(x,t)\partial_t\omega_i(x,t)dx \\ &= \int_0^{D_1} e^x\omega_i(x,t)\left(\partial_x\omega_i(x,t) - k_i^T e^{Ax}B\delta(t)\right)dx \\ &\quad + \int_{D_1}^{D_2} e^x\omega_i(x,t)\Big(\partial_x\omega_i(x,t) - k_i^T e^{A_1(x-D_1)}e^{AD_1}B\delta(t) \\ &\quad + \frac{k_i^T}{c_1+1}e^{A_1(x-D_1)}b_1k_1^Tp_1(D_1,t)\Big)dx \\ &\quad \vdots \\ &\quad + \int_{D_{i-1}}^{D_i} e^x\omega_i(x,t)\Big(\partial_x\omega_i(x,t) - k_i^T e^{A_{i-1}(x-D_{i-1})} \\ &\quad \times e^{A_{i-2}D_{i-1,i-2}}e^{A_{i-3}D_{i-2,i-3}}\cdots e^{A_1D_{2,1}}e^{AD_1}B\delta(t)\Big)dx \\ &\quad + \sum_{j=1}^{i-2}\int_{D_{i-1}}^{D_i}\Big(e^x\omega_i(x,t)k_i^Te^{A_{i-1}(x-D_{i-1})}e^{A_{i-2}D_{i-1,\,i-2}} \\ &\quad e^{A_{i-3}D_{i-2,\,i-3}}\cdots e^{A_jD_{j+1,j}}\frac{b_jk_j^T}{c_j+1}p_j(D_j,t)\Big)dx \\ &\quad + \int_{D_{i-1}}^{D_i}\Big(e^x\omega_i(x,t)k_i^Te^{A_{i-1}(x-D_{i-1})} \\ &\quad \times\frac{b_{i-1}k_{i-1}^T}{c_{i-1}+1}p_{i-1}(D_{i-1},t)\Big)dx.\end{aligned} \tag{46}$$

We estimate the first term of the right-hand side of (46) as

$$\begin{aligned}&\int_0^{D_1} e^x\omega_i(x,t)(\partial_x\omega_i(x,t) - k_i^Te^{Ax}B\delta(t))dx \\ &\le \frac{1}{2}e^{D_1}\omega_i(D_1,t)^2 - \frac{1}{2}\omega_i(0,t)^2 - \frac{1}{4}\int_0^{D_1} e^x\omega_i(x,t)^2dx \\ &\quad + D_1e^{D_1}|k_i|^2e^{2|A|D_1}|B|^2|\delta(t)|^2.\end{aligned} \tag{47}$$

Similarly, for the second term of the right-hand side of (46), we have

$$\begin{aligned}&\int_{D_1}^{D_2} e^x\omega_i(x,t)\Big(\partial_x\omega_i(x,t) - k_i^Te^{A_1(x-D_1)}e^{AD_1}B\delta(t) \\ &\quad + \frac{k_i^T}{c_1+1}e^{A_1(x-D_1)}b_1k_1^Tp_1(D_1,t)\Big)dx \\ &\le \frac{1}{2}e^{D_2}\omega_i(D_2,t)^2 - \frac{1}{2}e^{D_1}\omega_i(D_1,t)^2 - \frac{1}{4}\int_{D_1}^{D_2} e^x\omega_i(x,t)^2dx \\ &\quad + 2D_{2,1}e^{D_2}|k_i|^2e^{2|A_1|D_2}e^{2|A|D_1}|B|^2|\delta(t)|^2 \\ &\quad + 2D_{2,1}e^{D_2}\frac{|k_i|^2}{(c_1+1)^2}e^{2|A_1|D_2}|b_1|^2|k_1|^2|p_1(D_1,t)|^2.\end{aligned} \tag{48}$$

For the general lth term of (46), we get

$$\begin{aligned}\Gamma_l &= \int_{D_{l-1}}^{D_l} e^x\omega_i(x,t)\Big(\partial_x\omega_i(x,t) - k_i^Te^{A_{l-1}(x-D_{l-1})} \\ &\quad \times e^{A_{l-2}D_{l-1,l-2}}e^{A_{l-3}D_{l-2,l-3}}\cdots e^{A_1D_{2,1}}e^{AD_1}B\delta(t)\Big)dx \\ &\quad + \sum_{j=1}^{l-2}\int_{D_{l-1}}^{D_l}\Big(e^x\omega_i(x,t)k_i^Te^{A_{l-1}(x-D_{l-1})}e^{A_{l-2}D_{l-1,l-2}} \\ &\quad \times e^{A_{l-3}D_{l-2,l-3}}\cdots e^{A_jD_{j+1,j}}\frac{b_jk_j^T}{c_j+1}p_j(D_j,t)\Big)dx +\end{aligned}$$

$$\begin{aligned}
&\int_{D_{l-1}}^{D_l} e^x \omega_i(x,t) k_i^T e^{A_{l-1}(x-D_{l-1})} \frac{b_{l-1}k_{l-1}^T}{c_{l-1}+1} p_{l-1}(D_{l-1},t)dx \\
&\leq \frac{1}{2}e^{D_l}\omega_i(D_l,t)^2 - \frac{1}{2}e^{D_{l-1}}\omega_i(D_{l-1},t)^2 \\
&\quad -\frac{1}{4}\int_{D_{l-1}}^{D_l} e^x\omega_i(x,t)^2dx + lD_{l,l-1}e^{D_l}|k_i|^2 e^{2|A_{l-1}|D_{l,l-1}} \\
&\quad \times e^{2|A_{l-2}|D_{l-1,l-2}}\cdots e^{2|A|D_1}|B|^2|\delta(t)|^2 \\
&\quad + \sum_{j=1}^{l-2}\Big(lD_{l,l-1}e^{D_l}|k_i|^2 e^{2|A_{l-1}|D_{l,l-1}} e^{2|A_{l-2}|D_{l-1,l-2}}\cdots \\
&\quad \times e^{2|A_j|D_{j+1,j}}\frac{|b_j|^2|k_j|^2}{(c_j+1)^2}|p_j(D_j,t)|^2\Big) + lD_{l,l-1}e^{D_l}|k_i|^2 \\
&\quad \times e^{2|A_{l-1}|D_{l,l-1}}\frac{|b_{l-1}|^2|k_{l-1}|^2}{(c_{l-1}+1)^2}|p_{l-1}(D_{l-1},t)|^2, \qquad (49)
\end{aligned}$$

for all $l = 3, \ldots, i$. Recalling (30), from (47), (48), (49), we have

$$\begin{aligned}
&\int_0^{D_i} e^x\omega_i(x,t)\partial_t\omega_i(x,t)dx \\
&\leq \frac{1}{2}e^{D_i}\frac{1}{(c_i+1)^2}|k_i|^2|p_i(D_i,t)|^2 - \frac{1}{2}\omega_i(0,t)^2 \\
&\quad -\frac{1}{4}\int_0^{D_i} e^x\omega_i(x,t)^2dx + \varsigma_i|\delta(t)|^2 \\
&\quad +2D_{2,1}e^{D_2}\frac{|k_i|^2}{(c_1+1)^2}e^{2|A_1|D_2}|b_1|^2|k_1|^2|p_1(D_1,t)|^2 + \cdots \\
&\quad +lD_{l,l-1}e^{D_l}|k_i|^2\sum_{j=1}^{l-1}\Big(e^{2|A_{l-1}|D_{l,l-1}}e^{2|A_{l-2}|D_{l-1,l-2}}\cdots \\
&\quad \times e^{2|A_j|D_{j+1,j}}\frac{|b_j|^2|k_j|^2}{(c_j+1)^2}|p_j(D_j,t)|^2\Big) + \cdots \\
&\quad +iD_{i,i-1}e^{D_i}|k_i|^2\sum_{j=1}^{i-1}\Big(e^{2|A_{i-1}|D_{i,i-1}}e^{2|A_{i-2}|D_{i-1,i-2}}\cdots \\
&\quad \times e^{2|A_j|D_{j+1,j}}\frac{|b_j|^2|k_j|^2}{(c_j+1)^2}|p_j(D_j,t)|^2\Big) \qquad (50)
\end{aligned}$$

where

$$\begin{aligned}
\varsigma_i &= D_1e^{D_1}|k_i|^2e^{2|A|D_1}|B|^2 + 2D_{2,1}e^{D_2}|k_i|^2e^{2|A1|D_2}e^{2|A|D_1}|B|^2 \\
&\quad + \cdots + iD_{i,i-1}e^{D_i}|k_i|^2e^{2|A_{i-1}|D_{i,i-1}} \\
&\quad \times e^{2|A_{i-2}|D_{i-1,i-2}}\cdots e^{2|A|D_1}|B|^2. \qquad (51)
\end{aligned}$$

Denoting

$$\underline{c} = \min_{i=1,2,\ldots,m}\{c_i\}. \qquad (52)$$

$$\begin{aligned}
\rho_i &= \frac{1}{2}e^{D_i}|k_i|^2\gamma_i + 2D_{2,1}e^{D_2}|k_i|^2e^{2|A_1|D_2}|b_1|^2|k_1|^2|\gamma_1 \\
&\quad +lD_{l,l-1}e^{D_l}|k_i|^2\sum_{j=1}^{l-1}e^{2|A_{l-1}|D_{l,l-1}}e^{2|A_{l-2}|D_{l-1,l-2}}\cdots \\
&\quad \times e^{2|A_j|D_{j+1,j}}|b_j|^2|k_j|^2\gamma_j + \cdots \\
&\quad +iD_{i,i-1}e^{D_i}|k_i|^2\sum_{j=1}^{i-1}e^{2|A_{i-1}|D_{i,i-1}}e^{2|A_{i-2}|D_{i-1,i-2}}\cdots \\
&\quad \times e^{2|A_j|D_{j+1,j}}|b_j|^2|k_j|^2\gamma_j, \qquad (53)
\end{aligned}$$

with the help of (37), (42), (50), we finally get

$$\begin{aligned}
&\int_0^{D_i} e^x\omega_i(x,t)\partial_t\omega_i(x,t)dx \\
&\leq \frac{\alpha_2}{(\underline{c}+1)^2}\rho_i\left(|X(t)|^2 + \sum_{i=1}^m\|\omega_i(t)\|^2\right) - \frac{1}{2}\omega_i(0,t)^2 \\
&\quad -\frac{1}{4}\int_0^{D_i}e^x\omega_i(x,t)^2dx + \varsigma_i|\delta(t)|^2, \qquad (54)
\end{aligned}$$

for all $i = 1, 2, \ldots, m$. With (54), it can be deduced from (45) that

$$\begin{aligned}
\dot{V}(t) &\leq -\frac{\lambda_{\min}(S)}{2}X^T(t)X(t) \\
&\quad +\frac{4m\lambda_{\max}(M^2)}{\lambda_{\min}(S)}\max_{i=1,2,\ldots,m}\left\{|b_i|^2\right\}\sum_{i=1}^m\omega_i(0,t)^2 \\
&\quad +\frac{4\lambda_{\max}(MBB^TM)}{\lambda_{\min}(S)}|\delta(t)|^2 \\
&\quad +\frac{\alpha_2a_1}{(\underline{c}+1)^2}\left(|X(t)|^2+\sum_{i=1}^m\|\omega_i(t)\|^2\right)\sum_{i=1}^m\rho_i \\
&\quad -\frac{1}{2}a_1\sum_{i=1}^m\omega_i(0,t)^2 - \frac{a_1}{4}\sum_{i=1}^m\int_0^{D_i}e^x\omega_i(x,t)^2dx \\
&\quad +a_1|\delta(t)|^2\sum_{i=1}^m\varsigma_i. \qquad (55)
\end{aligned}$$

Let

$$a_1 = \frac{8m\lambda_{\max}(M^2)}{\lambda_{\min}(S)}\max_{i=1,2,\ldots,m}\left\{|b_i|^2\right\}+1. \qquad (56)$$

With (55), we get

$$\begin{aligned}
\dot{V}(t) &\leq -\left(\frac{\lambda_{\min}(S)}{2} - \frac{\alpha_2a_1}{(\underline{c}+1)^2}\sum_{i=1}^m\rho_i\right)|X(t)|^2 \\
&\quad -\left(\frac{a_1}{4} - \frac{\alpha_2a_1}{(\underline{c}+1)^2}\sum_{i=1}^m\rho_i\right)\sum_{i=1}^m\|\omega_i(t)\|^2 \\
&\quad +\left(\frac{4\lambda_{\max}(MBB^TM)}{\lambda_{\min}(S)}+a_1\sum_{i=1}^m\varsigma_i\right)|\delta(t)|^2. \qquad (57)
\end{aligned}$$

For $\underline{c} > c^*$, where

$$c^* = \frac{\sqrt{2\alpha_2\sum_{i=1}^m\rho_i\max\left\{\frac{a_1}{\lambda_{\min}(S)},2\right\}}}{\sqrt{1-\overline{\mu}}}, \qquad (58)$$

for some $0 < \overline{\mu} < 1$, we get

$$\begin{aligned}
\dot{V}(t) &\leq -\overline{\mu}\min\left\{\frac{\lambda_{\min}(S)}{2},\frac{a_1}{4}\right\}\left(|X(t)|^2+\sum_{i=1}^m\|\omega_i(t)\|^2\right) \\
&\quad +\left(\frac{4\lambda_{\max}(MBB^TM)}{\lambda_{\min}(S)}+a_1\sum_{i=1}^m\varsigma_i\right)|\delta(t)|^2. \qquad (59)
\end{aligned}$$

Moreover, from (44), we have

$$\begin{aligned}
&\min\left\{\lambda_{\min}(M),\frac{a_1}{2}\right\}\left(|X(t)|^2+\sum_{i=1}^m\|\omega_i(t)\|^2\right) \\
&\leq V(t) \\
&\leq \max\left\{\lambda_{\max}(M),\frac{a_1e^{D_m}}{2}\right\}\left(|X(t)|^2+\sum_{i=1}^m\|\omega_i(t)\|^2\right), \qquad (60)
\end{aligned}$$

and thus, from (59), (60), it holds that

$$\dot{V}(t) \le -\overline{\lambda} V(t) + \overline{\nu}|\delta(t)|^2, \tag{61}$$

with

$$\overline{\lambda} = \frac{\overline{\mu}\min\left\{\frac{\lambda_{\min}(S)}{2}, \frac{a_1}{2}\right\}}{\max\left\{\lambda_{\max}(M), \frac{a_1 e^{D_m}}{2}\right\}}, \tag{62}$$

$$\overline{\nu} = \frac{4\lambda_{\max}(MBB^TM)}{\lambda_{\min}(S)} + a_1\sum_{i=1}^m \varsigma_i. \tag{63}$$

Combining (15), (41), (42), (60), and (61), we get (9) with $\zeta = \frac{\alpha_2\overline{\nu}}{\overline{\lambda}\min\left\{\lambda_{\min}(M), \frac{a_1}{2}\right\}}$, $\psi = \frac{\alpha_1\alpha_2\max\left\{\lambda_{\max}(M), \frac{a_1e^{D_m}}{2}\right\}}{\min\left\{\lambda_{\min}(M), \frac{a_1}{2}\right\}}$.

3.2. Inverse optimality of the basic predictor-feedback controller

Theorem 2. *Consider system (1) together with the control laws (3)–(7). There exist $c^{**} \ge c^*$ and $d^{**} > 0$, such that for all $\underline{c} > c^{**}$ and $\underline{d} > d^{**}$, the control laws (3)–(7) minimize the cost functional*

$$J = \sup_{\delta\in\Xi}\lim_{t\to\infty}\left(2\beta V(t) + \int_0^t\left(L(\tau) + a_1\beta\sum_{i=1}^m \frac{e^{D_i}U_i(\tau)^2}{c_i}\right.\right.$$
$$\left.\left. -\underline{d}\beta|\delta(\tau)|^2\right)d\tau\right), \tag{64}$$

where L is a functional of $(X(t), U_1(\theta_1), \dots, U_m(\theta_m))$, $t - D_i \le \theta_i \le t$, $i = 1, \dots, m$, such that

$$L(t) \ge \beta\chi\Omega(t), \tag{65}$$

for an arbitrary $\beta > 0$ and some $\chi > 0$, and where a_1, V, Ω are given by (56), (44), (10), respectively, with Ξ being the set of l-dimensional vector-valued linear bounded functionals of $(X(t), U_1(\theta_1), \dots, U_m(\theta_m))$, $t - D_i \le \theta_i \le t$, $i = 1, \dots, m$.

Remark 4. Although cost functional (64) is not as general as a respective cost functional that would be employed in a direct optimal control approach, it is a meaningful cost since it puts quadratic penalties both on the control efforts and the disturbances, as well as on the overall infinite-dimensional state of the system (via the term L, which is lower bounded by Ω), and it also incorporates a terminal penalty. Moreover, the (inverse) optimality result in Theorem 2, is derived without needing to solve complicated operator Riccati equations and it provides an optimal value function that is actually a Lyapunov functional for the closed-loop system. Finally, note that inverse optimality also implies certain gain margin guarantees as it is evident in the present case from relation (3), which may be seen as a perturbed version of the nominal controller (4).

Proof of Theorem 2. Denote

$$\begin{aligned}\Theta_i(t) = &\int_{D_1}^{D_2} e^x\omega_i(x,t)\frac{k_i^T}{c_1+1}e^{A_1(x-D_1)}b_1k_1^Tp_1(D_1,t)dx\\ &+\cdots\\ &+\int_{D_{i-1}}^{D_i}\sum_{j=1}^{i-2}\left(e^x\omega_i(x,t)k_i^Te^{A_{i-1}(x-D_{i-1})}e^{A_{i-2}D_{i-1,i-2}}\right.\\ &\left.\times e^{A_{i-3}D_{i-2,i-3}}\cdots e^{A_jD_{j+1,j}}\frac{b_jk_j^T}{c_j+1}p_j(D_j,t)\right)dx\\ &+\int_{D_{i-1}}^{D_i} e^x\omega_i(x,t)k_i^Te^{A_{i-1}(x-D_{i-1})}\\ &\times\frac{b_{i-1}k_{i-1}^T}{c_{i-1}+1}p_{i-1}(D_{i-1},t)dx,\end{aligned} \tag{66}$$

for $i = 2, \dots, m$, and

$$\begin{aligned}\eta_i(t) = &-\int_0^{D_1} e^x\omega_i(x,t)k_i^Te^{Ax}dx\\ &-\int_{D_1}^{D_2} e^x\omega_i(x,t)k_i^Te^{A_1(x-D_1)}e^{AD_1}dx - \cdots\\ &-\int_{D_{i-1}}^{D_i} e^x\omega_i(x,t)k_i^Te^{A_{i-1}(x-D_{i-1})}e^{A_{i-2}D_{i-1,i-2}}\\ &\times e^{A_{i-3}D_{i-2,i-3}}\cdots e^{A_1D_{2,1}}e^{AD_1}dx,\end{aligned} \tag{67}$$

for $i = 1, 2, \dots, m$. Choose

$$\begin{aligned}L(t) = &-a_1\beta\sum_{i=1}^m\frac{e^{D_i}}{c_i+1}\overline{U}_i(t)^2 + a_1\beta\sum_{i=1}^m\omega_i(0,t)^2\\ &+a_1\beta\sum_{i=1}^m\int_0^{D_i}e^x\omega_i(x,t)^2dx + 2\beta X^T(t)SX(t)\\ &-4\beta X^T(t)M\sum_{i=1}^m b_i\omega_i(0,t) - 2a_1\beta\sum_{i=1}^m\Theta_i(t)\\ &-\frac{\beta}{\underline{d}}\left|2X^T(t)MB + a_1\sum_{i=1}^m\eta_i(t)B\right|^2,\end{aligned} \tag{68}$$

where a_1, $\overline{U}_i$, S are given by (56), (4), (43), respectively, and $\underline{d} > 0$, and β is an arbitrary positive scalar. From (66), (67), using Cauchy–Schwarz inequality, after some calculations, we have

$$\begin{aligned}\Theta_i(t) \le &\frac{1}{8}\int_0^{D_i}e^x\omega_i(x,t)^2dx\\ &+\frac{\alpha_2}{(\underline{c}+1)^2}\rho_i\left(|X(t)|^2 + \sum_{i=1}^m\|\omega_i(t)\|^2\right),\end{aligned} \tag{69}$$

for $i = 1, 2, \dots, m$, where $\underline{c}$, ρ_i, α_2 are given by (52), (53), (42), respectively, and

$$\begin{aligned}|\eta_i(t)|^2 \le\; &ie^{2D_1}\int_0^{D_1}\omega_i(x,t)^2dx\int_0^{D_1}\left|k_i^Te^{Ax}\right|^2dx\\ &+ie^{2D_{2,1}}\int_{D_1}^{D_2}\omega_i(x,t)^2dx\int_{D_1}^{D_2}\left|k_i^Te^{A_1(x-D_1)}e^{AD_1}\right|^2dx\\ &+\cdots\\ &+ie^{2D_{i,i-1}}\int_{D_{i-1}}^{D_i}\omega_i(x,t)^2dx\int_{D_{i-1}}^{D_i}\left|k_i^Te^{A_{i-1}(x-D_{i-1})}\right.\\ &\left.\times e^{A_{i-2}D_{i-1,i-2}}e^{A_{i-3}D_{i-2,i-3}}\cdots e^{A_1D_{2,1}}e^{AD_1}\right|^2dx\\ \le\; &\Lambda_i\int_0^{D_i}\omega_i(x,t)^2dx,\end{aligned} \tag{70}$$

where

$$\begin{aligned}\Lambda_i = &\max\{D_1e^{2D_1}, D_{2,1}e^{2D_{2,1}}, \dots, D_{i,i-1}e^{2D_{i,i-1}}\}\\ &\times i|k_i|^2e^{2\sum_{j=0}^{i-1}|A_j|D_{j+1,j}},\end{aligned} \tag{71}$$

for $i = 1, 2, \dots, m$.

Noting from (56) that $a_1 > \frac{8m\lambda_{\max}(M^2)}{\lambda_{\min}(S)}\max_{i=1,2,\dots,m}|b_i|^2$, by (4), (37), (42), (67)–(71), after some tedious calculations, we get

$$\begin{aligned}L(t) \ge \beta&\left(\frac{3\lambda_{\min}(S)}{2} - \frac{2a_1\alpha_2}{(\underline{c}+1)^2}\sum_{i=1}^m\rho_i\right.\\ &\left.-a_1\alpha_2\sum_{i=1}^m\frac{\gamma_ie^{D_i}}{c_i+1} - \frac{8}{\underline{d}}\lambda_{\max}(MBB^TM)\right)|X(t)|^2\end{aligned}$$

$$+\beta\left(\frac{3a_1}{4}-\frac{2a_1\alpha_2}{(\underline{c}+1)^2}\sum_{i=1}^{m}\rho_i -a_1\alpha_2\sum_{i=1}^{m}\frac{\gamma_i e^{D_i}}{c_i+1}-\frac{2a_1^2|B|^2m\xi}{\underline{d}}\right)\sum_{i=1}^{m}\|\omega_i(t)\|^2, \tag{72}$$

with $\xi=\max\{\Lambda_1,\dots,\Lambda_m\}$. Choose c^{**} and d^{**} such that

$$c^{**}\geq\max\left\{a_1\alpha_2\max\left\{\frac{2}{\lambda_{\min}(S)},\frac{4}{a_1}\right\}\times\left(2\sum_{i=1}^{m}\rho_i+\sum_{i=1}^{m}\gamma_i e^{Di}\right),\ c^*\right\}, \tag{73}$$

where c^* is defined in (58), and

$$d^{**}\geq\max\left\{\frac{16\lambda_{\max}(MBB^TM)}{\lambda_{\min}(S)},\ 8a_1|B|^2m\xi\right\}. \tag{74}$$

By (15), (42), (73) and (74), we get from (72) that

$$L(t)\geq\frac{\beta\min\left\{\frac{\lambda_{\min}(S)}{2},\frac{a_1}{4}\right\}\left(|X(t)|^2+\sum_{i=1}^{m}\|U_i(t)\|^2\right)}{\alpha_2}, \tag{75}$$

and hence, (65) is achieved with $\chi=\frac{\min\left\{\frac{\lambda_{\min}(S)}{2},\frac{a_1}{4}\right\}}{\alpha_2}$. With the help of (45), (46) and using (66), (67), from (68), we have

$$\begin{aligned}L(t)={}&-a_1\beta\sum_{i=1}^{m}\frac{e^{D_i}}{c_i+1}\overline{U}_i(t)^2+a_1\beta\sum_{i=1}^{m}e^{D_i}\omega_i(D_i,t)^2\\&-2\beta\dot{V}(t)+4\beta X^T(t)MB\delta(t)+2a_1\beta\sum_{i=1}^{m}\eta_i(t)B\delta(t)\\&-\frac{\beta}{\underline{d}}\left|2X^T(t)MB+a_1\sum_{i=1}^{m}\eta_i(t)B\right|^2.\end{aligned} \tag{76}$$

Furthermore, using the fact that $\omega_i(D_i,t)=U_i(t)-\overline{U}_i(t)$, for all $i=1,2,\dots,m$, and relation (3) we get

$$\begin{aligned}L(t)={}&a_1\beta\sum_{i=1}^{m}e^{D_i}\left(U_i(t)-U_i^*(t)\right)^2\\&-a_1\beta\sum_{i=1}^{m}e^{D_i}\frac{2U_i(t)U_i^*(t)}{c_i}+a_1\beta\sum_{i=1}^{m}e^{D_i}\frac{U_i^*(t)^2}{c_i}\\&-2\beta\dot{V}(t)+4\beta X^T(t)MB\delta(t)+2a_1\beta\sum_{i=1}^{m}\eta_i(t)B\delta(t)\\&-\frac{\beta}{\underline{d}}\left|2X^T(t)MB+a_1\sum_{i=1}^{m}\eta_i(t)B\right|^2.\end{aligned} \tag{77}$$

Denoting

$$\begin{aligned}\Pi(\delta(\tau))={}&4\beta X^T(\tau)MB\delta(\tau)+2a_1\beta\sum_{i=1}^{m}\eta_i(\tau)B\delta(\tau)\\&-\frac{\beta}{\underline{d}}\left|2X^T(\tau)MB+a_1\sum_{i=1}^{m}\eta_i(\tau)B\right|^2\\&-\underline{d}\beta|\delta(\tau)|^2,\end{aligned} \tag{78}$$

by (77), (78), completing the squares, it can be deduced that

$$\begin{aligned}&\int_0^t\left(L(\tau)+a_1\beta\sum_{i=1}^{m}\frac{e^{D_i}U_i(\tau)^2}{c_i}-\underline{d}\beta|\delta(\tau)|^2\right)d\tau\\&=-2\beta V(t)+2\beta V(0)\\&+a_1\beta\int_0^t\sum_{i=1}^{m}e^{D_i}\left(1+\frac{1}{c_i}\right)(U_i(\tau)-U_i^*(\tau))^2d\tau\\&+\int_0^t\Pi(\delta(\tau))d\tau.\end{aligned} \tag{79}$$

With the help of (79), we get from (64) that

$$\begin{aligned}J={}&2\beta V(0)+a_1\beta\int_0^\infty\sum_{i=1}^{m}e^{D_i}\left(1+\frac{1}{c_i}\right)(U_i(\tau)-U_i^*(\tau))^2d\tau\\&+\sup_{\delta\in\Xi}\int_0^\infty\Pi(\delta(\tau))d\tau.\end{aligned} \tag{80}$$

With (78), it can then be deduced that

$$\begin{aligned}\Pi(\delta(\tau))={}&-\beta\left|\frac{1}{\sqrt{\underline{d}}}\left(2X^T(\tau)M+a_1\sum_{i=1}^{m}\eta_i(\tau)\right)B\right.\\&\left.-\sqrt{\underline{d}}\delta^T(\tau)\right|^2\leq 0,\end{aligned} \tag{81}$$

with $\Pi(\delta)=0$, if and only if $\delta=\delta^*$, where

$$\delta^*=\frac{1}{\underline{d}}B^T\left(2M^TX+a_1\sum_{i=1}^{m}\eta_i^T\right). \tag{82}$$

Thus,

$$\sup_{\delta\in\Xi}\int_0^\infty\Pi(\delta(\tau))d\tau=0, \tag{83}$$

and the 'worst case' disturbance is given by (82). With (80) and (83), we get

$$J=2\beta V(0)+a_1\beta\int_0^\infty\sum_{i=1}^{m}e^{D_i}\left(1+\tfrac{1}{c_i}\right)(U_i(\tau)-U_i^*(\tau))^2d\tau. \tag{84}$$

So the minimum of (84) is reached with

$$U_i(t)=U_i^*(t), \tag{85}$$

for $i=1,2,\dots,m$, and is such that

$$J=2\beta V(0). \tag{86}$$

4. Example

Consider system (1) with the matrices A, b_1, b_2, and B given by

$$A=\begin{pmatrix}0&1&0\\-3&4&0\\-6&2&3\end{pmatrix},\ b_1=\begin{pmatrix}0\\1\\0\end{pmatrix},\ b_2=\begin{pmatrix}0\\-1\\1\end{pmatrix},\ B=\begin{pmatrix}0\\2\\3\end{pmatrix}. \tag{87}$$

It is easy to see that (A,b_1,b_2) is controllable, but neither (A,b_1) nor (A,b_2) alone are controllable. The nominal gains k_1,k_2 are (see (Tsubakino et al., 2016)

$$k_1=\begin{pmatrix}4&-10&0\end{pmatrix}^T,\ k_2=\begin{pmatrix}6&-2&-6\end{pmatrix}^T, \tag{88}$$

which render $A+b_1k_1^T+b_2k_2^T$ Hurwitz. Assume that there are delays $D_1=0.2$ and $D_2=0.5$ in the control inputs U_1 and U_2, respectively. The proposed control laws are

$$U_1(t)=\frac{c_1}{c_1+1}\begin{pmatrix}4&-10&0\end{pmatrix}P_1(t), \tag{89}$$

$$U_2(t)=\frac{c_2}{c_2+1}\begin{pmatrix}6&-2&-6\end{pmatrix}P_2(t), \tag{90}$$

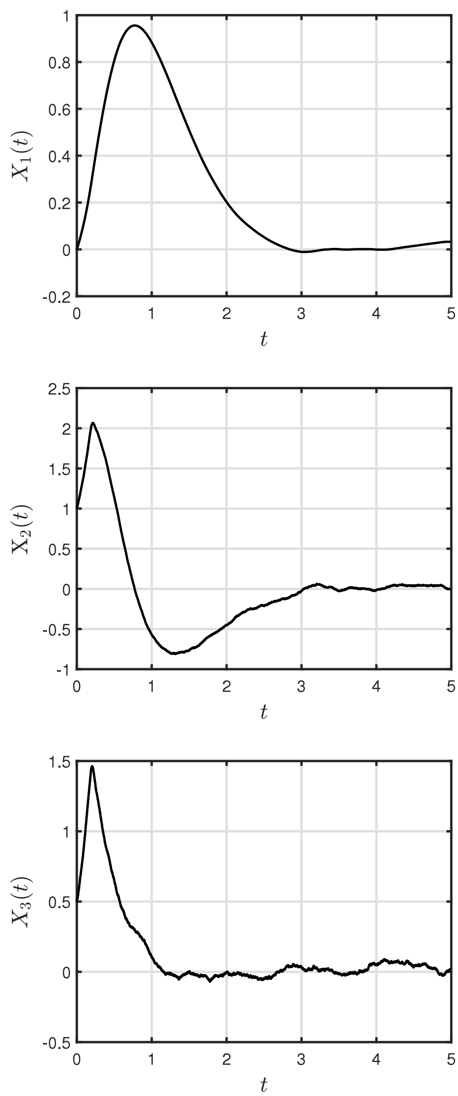

Fig. 1. Response of the states X_1, X_2, X_3 with the control laws (89)–(92) for initial conditions as $X_1(0) = 0$, $X_2(0) = 1$, $X_3(0) = 0.5$, and $U_1(\theta) = 0$, for $\theta \in [-0.2, 0]$, $U_2(\theta) = 0$, for $\theta \in [-0.5, 0]$.

where $c_i > 0, i = 1, 2$, are sufficiently large, and $P_i(t), i = 1, 2$, are given as

$$P_1(t) = e^{AD_1}X(t) + \int_{t-D_1}^{t} e^{A(t-s)}(b_1U_1(s) + b_2U_2(s - D_2 + D_1))ds, \tag{91}$$

$$P_2(t) = e^{A_1(D_2-D_1)}P_1(t) + \int_{t-D_2+D_1}^{t} e^{A_1(t-s)}b_2U_2(s)ds, \tag{92}$$

with $A_1 = A+b_1k_1^T$. The obtained allowable lower bound for c_1, c_2, within Theorem 1, may be somewhat conservative, yet, it may be computed explicitly using (58) as $c^* = 904.6266$, with $\overline{\mu} = 0.1$ and $S = 10I$.

Responses of the states under the control laws (89)–(92) are shown for $c_1 = c_2 = 1000$ in Fig. 1, whereas the control efforts are shown in Fig. 2. Disturbance $\delta(t)$ in Fig. 1 comprised randomly generated numbers from a uniform distribution in $[-1, 1]$. The closed-loop system is ISS.

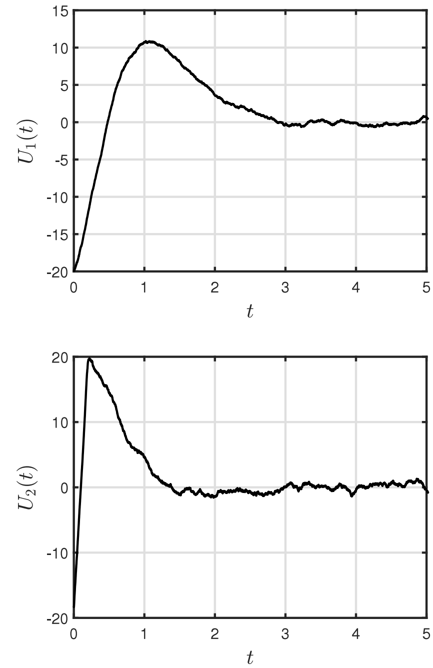

Fig. 2. Control laws (89)–(92) for $c_1 = c_2 = 1000$.

5. Conclusions

We consider multi-input linear systems, with distinct input delays in each individual input channel, under the predictor-feedback controller from Tsubakino et al. (2016). We established (1) ISS with respect to additive plant disturbances, (2) robustness to constant multiplicative perturbations appearing at the system inputs, and (3) inverse optimality with respect to a meaningful differential game problem. Our analyses are based on the availability of a backstepping transformation. Future research includes extensions to nonlinear systems as well as extensions to systems with more complex actuator dynamics than pure transport PDEs, with the results in Bekiaris-Liberis and Krstic (2011, 2014), Cai and Krstic (2015, 2016), as potential starting points.

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