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Simultaneous compensation of input and state delays for nonlinear systems

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ABSTRACT

The problem of compensation of arbitrary large input delay for nonlinear systems was solved recently with the introduction of the nonlinear predictor feedback. In this paper we solve the problem of compensation of input delay for nonlinear systems with simultaneous input and state delays of arbitrary length. The key challenge, in contrast to the case of only input delay, is that the input delay-free system (on which the design and stability proof of the closed-loop system under predictor feedback are based) is infinite-dimensional. We resolve this challenge and we design the predictor feedback law that compensates the input delay. We prove global asymptotic stability of the closed-loop system using two different techniques—one based on the construction of a Lyapunov functional, and one using estimates on solutions. We present two examples, one of a nonlinear delay system in the feedforward form with input delay, and one of a scalar, linear system with simultaneous input and state delays.

1. Introduction

Nonlinear delay systems are ubiquitous in applications. A non-exhaustive list includes traffic systems [1], additive manufacturing [2], oil drilling [3], automotive engines [4] and catalysts [5,6], energy systems, such as, for example, cooling systems [7], and networked control systems [8].

Nonlinear systems with state delays represent an advanced research area [9–15]. Numerous results also exist on the control and analysis of nonlinear systems with input delays [16–27]. Few results exist on the compensation of input delay for systems with simultaneous delay on the state, even for linear systems [28–30]. In [28] and [29] predictor feedback designs are developed, exploiting the special structure of the linear systems under consideration (systems in the feedback and feedforward form respectively), and in [30] a predictor feedback design is presented for general linear systems. Even fewer are the results dealing with the analysis and control of nonlinear systems with simultaneous input and state delays [31]. In [32] a predictor feedback law is developed for the compensation of input delay for a special class of nonlinear delay systems, namely, systems in the strict-feedback form with a state delay on the virtual input.

We consider nonlinear systems with simultaneous long discrete input delay and long (potentially distributed) state delay (the problem of the compensation of a distributed input delay is a different problem that goes beyond the predictor feedback approach that we present here, and therefore, we do not consider this problem in the present paper). We design a nonlinear predictor feedback law which employs, in a nominal feedback law that stabilizes the system with only the state delay, the predictor of the state over a prediction horizon equal to the length of the input delay, and hence, it achieves compensation of the input delay (Section 2). (Predictor feedback designs that also achieve compensation of the state delay, by exploiting the special structure of the system under consideration, are presented in [28] and in [32] for systems in the strict-feedback form with delays on the virtual inputs.) For nonlinear delay systems that are forward complete in the absence of the input delay we prove global asymptotic stability of the closed-loop system with the aid of a Lyapunov functional that we construct, based on the introduction of an infinite-dimensional backstepping transformation of the actuator state (Section 3).

We also present an alternative proof of global asymptotic stability by constructing estimates on the solutions of the closed-loop system and exploiting the facts that the input delay is compensated after a finite time-interval and that the system in the absence of only the input delay is forward complete (Section 4). We present a simulation example of a second-order nonlinear system in the strict-feedforward form with both input and state delays

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(Section 5). We also illustrate the linear case through an example of a scalar system with simultaneous input and state delays (Section 5).

Notation. We use the common definition of class $\mathcal{K}$, $\mathcal{K}_\infty$ and $\mathcal{KL}$ functions from [33]. For an n-vector, the norm $|\cdot|$ denotes the usual Euclidean norm. We denote by $C^j(A;\Omega)$ the class of functions, taking values in Ω, that have continuous derivatives of order j in A. We denote by $L^\infty(A;\Omega)$ the space of measurable and bounded functions defined on A and taking values in Ω. For a given $D_1 \geq 0$ and a function $\phi \in L^\infty([-D_1,0];\mathbb{R}^n)$ we denote by $\|\phi\|_{D_1}$ its supremum over $[-D_1,0]$, i.e., $\|\phi\|_{D_1} = \sup_{s\in[-D_1,0]}|\phi(s)|$. For a function $X:[-D_1,\infty)\to\mathbb{R}^n$, for all $t\geq 0$, the function X_t is defined by $X_t(s)=X(t+s)$, for all $s\in[-D_1,0]$. For a function $U:[-D_2,\infty)\to\mathbb{R}^n$, for all $t\geq 0$, the function U_t is defined by $U_t(s)=U(t+s)$, for all $s\in[-D_2,0]$. For a function $P:[-D_1-D_2,\infty)\to\mathbb{R}^n$, for all $\theta\geq -D_2$, the function P_θ is defined by $P_\theta(s)=P(\theta+s)$, for all $s\in[-D_1,0]$. Any relation in which the time t appears holds for all $t\geq 0$, unless stated otherwise.

2. Problem formulation and controller design

We consider the following system

$$\dot{X}(t) = f\left(X_t, U(t-D_2)\right), \tag{1}$$

for $t\geq 0$, where $f: C([-D_1,0];\mathbb{R}^n)\times\mathbb{R}\to\mathbb{R}^n$ is a locally Lipschitz mapping with $f(0,0)=0$, and $D_1, D_2\geq 0$. For designing a stabilizing feedback law for (1) one needs two ingredients. First, one needs a nominal feedback law that stabilizes system (1) when there is no input delay, i.e., system

$$\dot{X}(t) = f\left(X_t, U(t)\right). \tag{2}$$

The second ingredient one needs is the D_2-time units ahead predictor of X, that is, the signal P that satisfies $P(s)=X(s+D_2)$, for all $s\geq -D_1-D_2$. The controller that stabilizes system (1) and compensates the input delay is then given for $t\geq 0$ by

$$U(t) = \kappa\left(P_t\right), \tag{3}$$

where

$$P(\theta) = X(t) + \int_{t-D_2}^{\theta} f\left(P_s, U(s)\right) ds, \quad \text{for all } t-D_2\leq\theta\leq t, \tag{4}$$

with initial condition given by

$$P(s) = X\left(s+D_2\right), \quad \text{for all } -D_1-D_2\leq s\leq -D_2 \tag{5}$$

$$P(\theta) = X(0) + \int_{-D_2}^{\theta} f\left(P_\sigma, U(\sigma)\right) d\sigma, \quad \text{for all } -D_2\leq\theta\leq 0. \tag{6}$$

The fact that P is the D_2-time units ahead predictor of X can be seen as follows. Performing the change of variables $t=\theta+D_2$, for all $t-D_2\leq\theta\leq t$ in (1) and integrating starting at $\theta=t-D_2$ we get that

$$X\left(\theta+D_2\right) = X(t) + \int_{t-D_2}^{\theta} f\left(X_{s+D_2}, U(s)\right) ds. \tag{7}$$

Defining $P(\theta)=X(\theta+D_2)$, for all $t-D_2\leq\theta\leq t$ and using the fact that P satisfies (5), we conclude that the signal P defined by (4), with initial conditions (5), (6) satisfies $P(s)=X(s+D_2)$, for all $s\geq -D_1-D_2$.

3. Lyapunov-based stability analysis

Assumption 1. System (2) is forward complete.

Assumption 1 guarantees that for every initial condition $X_0\in C([-D_1,0];\mathbb{R}^n)$ and for every locally bounded input signal U the corresponding solution is defined for all $t\geq 0$.

Assumption 2. There exist a locally Lipschitz feedback law $\kappa: C([-D_1,0];\mathbb{R}^n)\to\mathbb{R}$ with $\kappa(0)=0$ and a class $\mathcal{K}_\infty$ function α such that for all $\phi\in C([-D_1,0];\mathbb{R}^n)$

$$|\kappa(\phi)| \leq \alpha\left(\|\phi\|_{D_1}\right), \tag{8}$$

a locally Lipschitz functional $S: C([-D_1,0];\mathbb{R}^n)\to\mathbb{R}_+$, and class $\mathcal{K}_\infty$ functions $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ such that for all $\phi\in C([-D_1,0];\mathbb{R}^n)$ it holds that

$$\alpha_1\left(|\phi(0)|\right) \leq S(\phi) \leq \alpha_2\left(\|\phi\|_{D_1}\right), \tag{9}$$

and along the trajectories of the closed-loop system $\dot{X}(t) = f\left(X_t, \kappa\left(X_t\right)+\omega(t)\right)$, S is continuously differentiable and satisfies for all $\omega\in C([0,+\infty);\mathbb{R})$

$$\dot{S}(t) \leq -\alpha_3\left(S\left(X_t\right)\right) + \alpha_4\left(|\omega(t)|\right), \tag{10}$$

for all $t\geq 0$.

Theorem 1. *Consider system (1) together with the control law (3)–(6). Under Assumptions 1 and 2 there exists a class $\mathcal{KL}$ function β such that for all initial conditions $X_0\in C([-D_1,0];\mathbb{R}^n)$ and $U_0\in C([-D_2,0];\mathbb{R})$, that are compatible with the feedback law, that is, they satisfy $U_0(0)=\kappa(P_0)$, there exists a unique solution to the closed-loop system with $X\in C^1([0,+\infty);\mathbb{R}^n)$, $U\in C([0,+\infty);\mathbb{R})$, and the following holds*

$$\sup_{t-D_1\leq\tau\leq t}|X(\tau)| + \sup_{t-D_2\leq\theta\leq t}|U(\theta)| \leq \beta\left(\sup_{-D_1\leq\tau\leq 0}|X(\tau)| + \sup_{-D_2\leq\theta\leq 0}|U(\theta)|, t\right), \tag{11}$$

for all $t\geq 0$.

The proof of Theorem 1 is based on a series of technical lemmas that are presented next.

Lemma 1. *The infinite-dimensional backstepping transformation of the actuator state defined by*

$$W(\theta) = U(\theta) - \kappa\left(P_\theta\right), \quad t-D_2\leq\theta\leq t, \tag{12}$$

together with the predictor feedback law (3)–(6) transform the system (1) to the "target system" given by

$$\dot{X}(t) = f\left(X_t, \kappa\left(X_t\right) + W\left(t-D_2\right)\right) \tag{13}$$

$$W(t) = 0, \quad \forall t\geq 0. \tag{14}$$

Proof. Using (1) and the fact that $P_{t-D_2}=X_t$ we get (13). With (3) we get (14).

Lemma 2. *The inverse of the infinite-dimensional backstepping transformation defined in (12) is given by*

$$U(\theta) = W(\theta) + \kappa\left(\Pi_\theta\right), \quad t-D_2\leq\theta\leq t, \tag{15}$$

where

$$\Pi(\theta) = X(t) + \int_{t-D_2}^{\theta} f\left(\Pi_s, \kappa\left(\Pi_s\right) + W(s)\right) ds, \quad \text{for all } t-D_2\leq\theta\leq t, \tag{16}$$

with initial condition given by

$$\Pi(s) = X(s+D_2), \quad \textit{for all } -D_1-D_2 \le s \le -D_2 \tag{17}$$

$$\Pi(\theta) = X(0) + \int_{-D_2}^{\theta} f\left(\Pi_\sigma, \kappa\left(\Pi_\sigma\right) + W(\sigma)\right) d\sigma, \quad \textit{for all } -D_2 \le \theta \le 0. \tag{18}$$

Proof. By direct verification, noting also that $\Pi_\theta = P_\theta$ for all $t - D_2 \le \theta \le t$, where $\Pi(\theta)$ is driven by the transformed input $W(\theta)$, whereas $P(\theta)$ is driven by the input $U(\theta)$.

Lemma 3. *There exists a class $\mathcal{KL}$ function β_1 such that the following holds*

$$\sup_{t-D_1\le\tau\le t}|X(\tau)| + \sup_{t-D_2\le\theta\le t}|W(\theta)| \le \beta_1\left(\sup_{-D_1\le\tau\le 0}|X(\tau)| + \sup_{-D_2\le\theta\le 0}|W(\theta)|, t\right), \tag{19}$$

for all $t \ge 0$.

Proof. Consider the following Lyapunov functional for the "target system" given in (13)–(14)

$$V(t) = S(t) + k\int_0^{L(t)} \frac{\alpha_4(r)}{r} dr, \tag{20}$$

where

$$L(t) = \sup_{t-D_2\le\theta\le t}\left|e^{c(\theta-t+D_2)}W(\theta)\right| = \lim_{n\to\infty}\left(\int_{t-D_2}^{t} e^{2nc(\theta-t+D_2)}W(\theta)^{2n}d\theta\right)^{\frac{1}{2n}}, \tag{21}$$

with an arbitrary $c > 0$, satisfies

$$L(t) \le e^{cD_2}\sup_{t-D_2\le\theta\le t}|W(\theta)| \tag{22}$$

$$L(t) \ge \sup_{t-D_2\le\theta\le t}|W(\theta)|. \tag{23}$$

Taking the time derivative of $L(t)$ and using (14) we get

$$\dot{L}(t) = \lim_{n\to\infty}\frac{1}{2n}\left(\int_{t-D_2}^{t} e^{2nc(\theta-t+D_2)}W(\theta)^{2n}d\theta\right)^{\frac{1}{2n}-1} \times\left(-W(t-D_2)^{2n} - 2nc\int_{t-D_2}^{t} e^{2nc(\theta-t+D_2)}W(\theta)^{2n}d\theta\right), \tag{24}$$

and hence, $\dot{L}(t) \le -cL(t)$. With this inequality and (10), taking the derivative of (20) we get

$$\dot{V}(t) \le -\alpha_3\left(S\left(X_t\right)\right) + \alpha_4\left(|W(t-D_2)|\right) - kc\alpha_4\left(L(t)\right). \tag{25}$$

With the help of (23) and choosing $k = \frac{2}{c}$ we get $\dot{V}(t) \le -\alpha_3(S(X_t)) - \alpha_4(L(t))$, and hence, with definition (20) we conclude that there exists a class $\mathcal{K}$ function γ_1 such that $\dot{V}(t) \le -\gamma_1(V(t))$. Using the comparison principle and Lemma 4.4 in [33], there exists a class $\mathcal{KL}$ function β_1 such that

$$V(t) \le \beta_1(V(0), t). \tag{26}$$

Using (9), the definition of $V(t)$ in (20) and the properties of class $\mathcal{K}$ functions we arrive at

$$|X(t)| + L(t) \le \beta_2\left(\sup_{-D_1\le\tau\le 0}|X(\tau)| + L(0), t\right) \tag{27}$$

for some class $\mathcal{KL}$ function β_2. Using relations (22) and (23) we get that there exists a class $\mathcal{KL}$ function β_3 such that

$$|X(t)| + \sup_{t-D_2\le\theta\le t}|W(\theta)| \le \beta_3\left(\sup_{-D_1\le\tau\le 0}|X(\tau)| + \sup_{-D_2\le\theta\le 0}|W(\theta)|, t\right), \tag{28}$$

for all $t \ge 0$. Therefore,

$$\sup_{t-D_1\le\tau\le t}|X(\tau)| \le \beta_3\left(\sup_{-D_1\le\tau\le 0}|X(\tau)| + \sup_{-D_2\le\theta\le 0}|W(\theta)|, t-D_1\right), \quad \text{for all } t \ge D_1. \tag{29}$$

With the fact that for all $t \le D_1$, $\sup_{t-D_1\le\tau\le t}|X(\tau)| \le \sup_{-D_1\le\tau\le 0}|X(\tau)| + \sup_{0\le\tau\le D_1}|X(\tau)|$ and using (28) we get that

$$\sup_{t-D_1\le\tau\le t}|X(\tau)| \le \alpha_5\left(\sup_{-D_1\le\tau\le 0}|X(\tau)| + \sup_{-D_2\le\theta\le 0}|W(\theta)|\right), \quad \text{for all } t \le D_1, \tag{30}$$

where the class $\mathcal{K}_\infty$ function α_5 is defined as $\alpha_5(s) = \beta_3(s, 0) + s$. Combining (29) and (30) we arrive at

$$\sup_{t-D_1\le\tau\le t}|X(\tau)| \le \beta_4\left(\sup_{-D_1\le\tau\le 0}|X(\tau)| + \sup_{-D_2\le\theta\le 0}|W(\theta)|, t\right), \quad \text{for all } t \ge 0, \tag{31}$$

where $\beta_4(s,t) = \alpha_5(s)e^{-\max\{0,t-D_1\}} + \beta_3(s, \max\{0, t-D_1\})$ is of class $\mathcal{KL}$. Using (28) and (31) the lemma is proved.

Lemma 4. *There exists a class $\mathcal{K}_\infty$ function α_6 such that the following holds for all $t \ge 0$*

$$\sup_{\theta-D_1\le\tau\le\theta}|P(\tau)| \le \alpha_6\left(\sup_{t-D_1\le\tau\le t}|X(\tau)| + \sup_{t-D_2\le\tau\le t}|U(\tau)|\right), \quad \textit{for all } t-D_2 \le \theta \le t. \tag{32}$$

Proof. Defining the state $Y^t(s) = X(t+s)$, for all $t \ge 0$, which is parametrized in t and viewing s as the running parameter we get using (1) that $Y^t(s)$ satisfies the following delay differential equation in s

$$Y^t(s)' = f\left({Y^t}_s, V^t(s-D_2)\right), \quad \text{for all } 0 \le s \le D_2, \tag{33}$$

with initial condition $Y^t(\tau) = X(t+\tau)$, for all $-D_1 \le \tau \le 0$, where $V^t(s-D_2) = U(t+s-D_2)$. Under Assumption 1 and [34] we get that there exist a class $\mathcal{K}$ function ψ and a continuous positive-valued monotonically increasing function ν such that

$$\sup_{s-D_1\le\tau\le s}|Y^t(\tau)| \le \nu(D_2)\psi\left(\sup_{-D_1\le\tau\le 0}|Y^t(\tau)| + \sup_{-D_2\le\tau\le s-D_2}|V^t(\tau)|\right), \quad \text{for all } 0 \le s \le D_2, \tag{34}$$

and hence, with definitions $Y^t(s) = X(t+s)$ and $V^t(s-D_2) = U(t+s-D_2)$, we arrive at

$$\sup_{t-D_1+s\le\tau\le t+s}|X(\tau)| \le \nu(D_2)\psi\left(\sup_{t-D_1\le\tau\le t}|X(\tau)| + \sup_{t-D_2\le\tau\le t+s-D_2}|U(\tau)|\right), \quad \text{for all } 0 \le s \le D_2. \tag{35}$$

With the fact that $P(t+\tau-D_2)=X(t+\tau)$, for all $-D_1 \le \tau \le D_2$ we get from (35) that

$$\sup_{t-D_1-D_2+s\le\tau\le t+s-D_2} |P(\tau)| \le \nu(D_2)\psi\left(\sup_{t-D_1\le\tau\le t}|X(\tau)|+\sup_{t-D_2\le\tau\le t}|U(\tau)|\right), \quad \text{for all } 0\le s\le D_2. \tag{36}$$

Therefore, by setting $\theta=s+t-D_2$, for all $t-D_2\le\theta\le t$ we get estimate (32).

Lemma 5. *There exists a class $\mathcal{K}_\infty$ function α_7 such that the following holds for all $t\ge 0$*

$$\sup_{\theta-D_1\le\tau\le\theta}|\Pi(\tau)|\le\alpha_7\left(\sup_{t-D_1\le\tau\le t}|X(\tau)|+\sup_{t-D_2\le\tau\le t}|W(\tau)|\right), \quad \text{for all } t-D_2\le\theta\le t, \tag{37}$$

where Π is defined in Lemma 2.

Proof. Similarly to the proof of Lemma 4 we get using (13) that $Y^t(s)$ satisfies the following delay differential equation in s for all $t\ge 0$

$$Y^t(s)'=f\left({Y^t}_s,\kappa\left({Y^t}_s\right)+\hat{V}^t(s-D_2)\right), \quad \text{for all } 0\le s\le D_2, \tag{38}$$

where $\hat{V}^t(s-D_2)=W(t+s-D_2)$. Using (10) and Corollary IV.3 in [35] one can conclude that there exists a class $\mathcal{KL}$ function β_5 such that

$$S\left({Y^t}_s\right)\le\beta_5\left(S\left({Y^t}_0\right),s\right)+2\int_0^s\alpha_4\left(\left|\hat{V}^t(\tau-D_2)\right|\right)d\tau, \quad \text{for all } 0\le s\le D_2. \tag{39}$$

Using (9) we obtain for all $0\le s\le D_2$,

$$\alpha_1\left(\left|Y^t(s)\right|\right)\le\beta_5\left(\alpha_2\left(\sup_{-D_1\le\tau\le 0}\left|Y^t(\tau)\right|\right),0\right)+2D_2\alpha_4\left(\sup_{0\le\tau\le D_2}\left|\hat{V}^t(\tau-D_2)\right|\right), \tag{40}$$

and hence, we arrive at

$$\left|Y^t(s)\right|\le\alpha_8\left(\sup_{-D_1\le\tau\le 0}|Y^t(\tau)|+\sup_{-D_2\le\theta\le 0}|\hat{V}^t(\theta)|\right), \quad \text{for all } 0\le s\le D_2, \tag{41}$$

for some class $\mathcal{K}_\infty$ function α_8. Using the fact that for all $0\le s\le D_2$ it holds that $\sup_{s-D_1\le\tau\le s}\left|Y^t(\tau)\right|\le\sup_{-D_1\le\tau\le D_2}\left|Y^t(\tau)\right|\le\sup_{-D_1\le\tau\le 0}\left|Y^t(\tau)\right|+\sup_{0\le\tau\le D_2}\left|Y^t(\tau)\right|$ and relation (41) we obtain

$$\sup_{s-D_1\le\tau\le s}\left|Y^t(\tau)\right|\le\alpha_9\left(\sup_{-D_1\le\tau\le 0}|Y^t(\tau)|+\sup_{-D_2\le\theta\le 0}|\hat{V}^t(\theta)|\right), \quad \text{for all } 0\le s\le D_2, \tag{42}$$

where $\alpha_9(s)=\alpha_8(s)+s$. Using definitions $Y^t(\tau)=X(t+\tau)$ and $\hat{V}^t(\tau-D_2)=W(t+\tau-D_2)$ we get

$$\sup_{t+s-D_1\le\tau\le t+s}|X(\tau)|\le\alpha_9\left(\sup_{t-D_1\le\tau\le t}|X(\tau)|+\sup_{t-D_2\le\theta\le t}|W(\theta)|\right), \quad \text{for all } 0\le s\le D_2. \tag{43}$$

Since $\Pi(\tau-D_2)=X(\tau)$, for all $\tau\ge -D_1$ we get that

$$\sup_{t+s-D_1-D_2\le\tau\le t+s-D_2}|\Pi(\tau)|\le\alpha_9\left(\sup_{t-D_1\le\tau\le t}|X(\tau)|+\sup_{t-D_2\le\theta\le t}|W(\theta)|\right), \quad \text{for all } 0\le s\le D_2, \tag{44}$$

and hence, by setting $\theta=s+t-D_2$, for all $t-D_2\le\theta\le t$ the lemma is proved.

Lemma 6. *There exist class $\mathcal{K}_\infty$ functions α_{10}, α_{11} such that the following holds*

$$\sup_{t-D_1\le\theta\le t}|X(\theta)|+\sup_{t-D_2\le\theta\le t}|U(\theta)| \le\alpha_{10}\left(\sup_{t-D_1\le\theta\le t}|X(\theta)|+\sup_{t-D_2\le\theta\le t}|W(\theta)|\right) \tag{45}$$

$$\sup_{t-D_1\le\theta\le t}|X(\theta)|+\sup_{t-D_2\le\theta\le t}|W(\theta)| \le\alpha_{11}\left(\sup_{t-D_1\le\theta\le t}|X(\theta)|+\sup_{t-D_2\le\theta\le t}|U(\theta)|\right), \tag{46}$$

for all $t\ge 0$, where W is defined in Lemma 1.

Proof. Under Assumption 2 (relation (8)) we get that

$$|\kappa(P_\theta)|\le\alpha\left(\sup_{\theta-D_1\le\tau\le\theta}|P(\tau)|\right), \quad \text{for all } t-D_2\le\theta\le t. \tag{47}$$

Using (8) in Assumption 2 and (15) we get

$$|U(\theta)|\le|W(\theta)|+\alpha\left(\sup_{\theta-D_1\le\tau\le\theta}|\Pi(\tau)|\right), \quad \text{for all } t-D_2\le\theta\le t, \tag{48}$$

and hence, since the right-hand side of estimate (37) does not depend on θ we get (45) with $\alpha_{10}(s)=s+\alpha\left(\alpha_7(s)\right)$. Analogously, combining (47), (12) and (32) we get (46) with $\alpha_{11}(s)=s+\alpha\left(\alpha_6(s)\right)$.

Proof of Theorem 1. Using (19) we get (11) with $\beta(s,t)=\alpha_{10}\left(\beta_1\left(\alpha_{11}(s),t\right)\right)$. With Theorem 3.2 in [36] and from (1), the fact that $U\in C\left([-D_2,0];\mathbb{R}\right)$ guarantees the existence and uniqueness of $X\in C^1[0,D_2)$, and the system (13), (14) guarantees the existence and uniqueness of $X\in C^1(D_2,\infty)$. The compatibility condition of the feedback law guarantees that X is continuously differentiable also at $t=D_2$. Since $U(t)=\kappa(P_t)=\kappa\left(X_{t+D_2}\right)$, the facts that $X\in C^1[0,\infty)$ and Assumption 2 (Lipschitzness of κ), guarantee that U is continuous on $[0,\infty)$ for $D_2\ge D_1$. When $D_2\le D_1$, the facts that $X\in C^1[0,\infty)$ and that $X_0\in C\left([-D_1,0];\mathbb{R}\right)$, and Assumption 2 (Lipschitzness of κ), guarantee that U is continuous on $[0,\infty)$. □

4. Stability analysis using solutions' estimates

Assumption 3. There exists a locally Lipschitz feedback law $\kappa: C\left([-D_1,0];\mathbb{R}^n\right)\to\mathbb{R}$ with $\kappa(0)=0$, that renders system (2) globally asymptotically stable, in the sense that there exists a class $\mathcal{KL}$ function $\hat{\beta}$ such that for all $X_0\in C\left([-D_1,0];\mathbb{R}^n\right)$ the solution $X(t)$ of (2) with $U(t)=\kappa\left(X_t\right)$ satisfies

$$|X(t)|\le\hat{\beta}\left(\sup_{-D_1\le\theta\le 0}|X(\theta)|,t\right), \tag{49}$$

for all $t\ge 0$.

We are now ready to state the following result.

Theorem 2. *Consider the plant (1) together with the control law (3)–(6). Under Assumptions 1 and 3 there exists a class* $\mathcal{KL}$ *function* β^* *such that for all initial conditions* $X_0 \in C([-D_1, 0]; \mathbb{R}^n)$ *and* $U_0 \in C([-D_2, 0]; \mathbb{R})$*, that are compatible with the feedback law, that is, they satisfy* $U_0(0) = \kappa(P_0)$*, there exists a unique solution to the closed-loop system with* $X \in C^1([0, +\infty); \mathbb{R}^n)$*,* $U \in C([0, +\infty); \mathbb{R})$*, and the following holds*

$$\sup_{t-D_1\le\theta\le t}|X(\theta)| + \sup_{t-D_2\le\theta\le t}|U(\theta)| \le \beta^*\left(\sup_{-D_1\le\theta\le 0}|X(\theta)| + \sup_{-D_2\le\theta\le 0}|U(\theta)|, t\right), \quad (50)$$

for all $t \ge 0$.

Proof. Under Assumption 1 and [34] we get that there exist a class $\mathcal{K}$ function ψ and a continuous positive-valued monotonically increasing function ν such that

$$\sup_{t-D_1\le\theta\le t}|X(\theta)| \le \nu(t)\psi\left(\sup_{-D_1\le\theta\le 0}|X(\theta)| + \sup_{-D_2\le\tau\le t-D_2}|U(\tau)|\right),$$
$$\text{for all } t \ge 0, \quad (51)$$

and hence,

$$\sup_{t-D_1\le\theta\le t}|X(\theta)| \le \nu(D_2)\psi\left(\sup_{-D_1\le\theta\le 0}|X(\theta)| + \sup_{-D_2\le\theta\le 0}|U(\theta)|\right),$$
$$\text{for all } t \le D_2. \quad (52)$$

Under Assumption 3 and using the fact that $U(t) = \kappa(P_t) = \kappa\left(X_{t+D_2}\right)$, for all $t \ge 0$, we get that there exists a class $\mathcal{KL}$ function $\hat{\beta}$ such that

$$|X(t)| \le \hat{\beta}\left(\sup_{D_2-D_1\le\theta\le D_2}|X(\theta)|, t - D_2\right), \quad \text{for all } t \ge D_2. \quad (53)$$

Therefore,

$$\sup_{t-D_1\le\theta\le t}|X(\theta)| \le \hat{\beta}\left(\sup_{D_2-D_1\le\theta\le D_2}|X(\theta)|, t - D_2 - D_1\right),$$
$$\text{for all } t \ge D_2 + D_1. \quad (54)$$

We estimate next $\sup_{t-D_1\le\theta\le t}|X(\theta)|$, for all $D_2 \le t \le D_1 + D_2$. It holds that

$$\sup_{t-D_1\le\theta\le t}|X(\theta)| \le \sup_{D_2-D_1\le\theta\le D_2}|X(\theta)| + \sup_{D_2\le\theta\le D_1+D_2}|X(\theta)|,$$
$$\text{for all } D_2 \le t \le D_1 + D_2, \quad (55)$$

and hence, using (52) and (53) we get that

$$\sup_{t-D_1\le\theta\le t}|X(\theta)| \le \hat{\alpha}_1\left(\sup_{-D_1\le\theta\le 0}|X(\theta)| + \sup_{-D_2\le\theta\le 0}|U(\theta)|\right),$$
$$\text{for all } t \le D_1 + D_2, \quad (56)$$

where the class $\mathcal{K}_\infty$ function $\hat{\alpha}_1$ is defined as $\hat{\alpha}_1(s) = s + \nu(D_2)\psi(s) + \hat{\beta}(\nu(D_2)\psi(s), 0)$ and we also used the fact that $\hat{\alpha}_1(s) \ge \nu(D_2)\psi(s)$. Combining (52) and (54) we get that

$$\sup_{t-D_1\le\theta\le t}|X(\theta)| \le \hat{\beta}_1\left(\sup_{-D_1\le\theta\le 0}|X(\theta)| + \sup_{-D_2\le\theta\le 0}|U(\theta)|, t - D_1 - D_2\right),$$
$$\text{for all } t \ge D_1 + D_2, \quad (57)$$

where the class $\mathcal{KL}$ function $\hat{\beta}_1$ is defined as $\hat{\beta}_1(s, t) = \hat{\beta}(\nu(D_2)\psi(s), t)$. Therefore, using (56), (57) we get that

$$\sup_{t-D_1\le\theta\le t}|X(\theta)| \le \hat{\beta}_2\left(\sup_{-D_1\le\theta\le 0}|X(\theta)| + \sup_{-D_2\le\theta\le 0}|U(\theta)|, t\right),$$
$$\text{for all } t \ge 0, \quad (58)$$

where $\hat{\beta}_2(s, t) = \hat{\alpha}_1(s)e^{-\max\{0, t-D_1-D_2\}} + \hat{\beta}_1(s, \max\{0, t - D_1 - D_2\})$ is a class $\mathcal{KL}$ function.

We estimate next $\sup_{t-D_2\le\theta\le t}|U(\theta)|$. Using (8), (56), (57), and the fact that $U(t) = \kappa(P_t) = \kappa\left(X_{t+D_2}\right)$ we get that

$$|U(t)| \le \alpha\left(\hat{\beta}_3\left(\sup_{-D_1\le\theta\le 0}|X(\theta)| + \sup_{-D_2\le\theta\le 0}|U(\theta)|, t\right)\right),$$
$$\text{for all } t \ge 0, \quad (59)$$

where $\hat{\beta}_3(s, t) = \hat{\alpha}_1(s)e^{-\max\{0, t-D_1\}} + \hat{\beta}_1(s, \max\{0, t - D_1\})$ is a class $\mathcal{KL}$ function. Therefore,

$$\sup_{t-D_2\le\theta\le t}|U(\theta)| \le \hat{\beta}_4\left(\sup_{-D_1\le\theta\le 0}|X(\theta)| + \sup_{-D_2\le\theta\le 0}|U(\theta)|, t - D_2\right),$$
$$\text{for all } t \ge D_2, \quad (60)$$

where $\hat{\beta}_4(s, t) = \alpha\left(\hat{\beta}_3(s, t)\right)$ is a class $\mathcal{KL}$ function. Since for all $t \le D_2$

$$\sup_{t-D_2\le\theta\le t}|U(\theta)| \le \sup_{-D_2\le\theta\le 0}|U(\theta)| + \sup_{0\le\theta\le D_2}|U(\theta)|, \quad (61)$$

we get from (59) that

$$\sup_{t-D_2\le\theta\le t}|U(\theta)| \le \hat{\alpha}_2\left(\sup_{-D_1\le\theta\le 0}|X(\theta)| + \sup_{-D_2\le\theta\le 0}|U(\theta)|\right),$$
$$\text{for all } t \le D_2, \quad (62)$$

where the class $\mathcal{K}_\infty$ function $\hat{\alpha}_2$ is defined as $\hat{\alpha}_2(s) = s + \alpha\left(\hat{\beta}_3(s, 0)\right)$. Combining (60), (62) we arrive at

$$\sup_{t-D_2\le\theta\le t}|U(\theta)| \le \hat{\beta}_5\left(\sup_{-D_1\le\theta\le 0}|X(\theta)| + \sup_{-D_2\le\theta\le 0}|U(\theta)|, t\right),$$
$$\text{for all } t \ge 0, \quad (63)$$

where $\hat{\beta}_5(s, t) = \hat{\alpha}_2(s)e^{-\max\{0, t-D_2\}} + \hat{\beta}_4(s, \max\{0, t - D_2\})$ is a class $\mathcal{KL}$ function. Combining estimates (58) and (63) we get (50) with

$$\beta^*(s, t) = \hat{\beta}_2(s, t) + \hat{\beta}_5(s, t). \quad (64)$$

The rest of the result follows as in the proof of Theorem 1.

5. Examples

Example 1. We consider the following system

$$\dot{X}_1(t) = X_2(t - D_1) + X_2(t)^2 \quad (65)$$
$$\dot{X}_2(t) = U(t - D_2). \quad (66)$$

For the case $D_2 = 0$ a control law that achieves global asymptotic stabilization is constructed in [10] as

$$U(t) = -2X_2(t) - (1 + X_2(t)) \times \left(X_1(t) + X_2(t) + \frac{1}{2}X_2(t)^2 + \int_{t-D_1}^{t} X_2(\theta)d\theta\right). \quad (67)$$

Using Theorem 2.2 in [37] one can conclude that estimate (49) holds, and hence, Theorem 2 can be applied to system (65), (66) with the nominal control law (67). The predictor feedback law that globally asymptotically stabilizes (65), (66) is given by

$$U(t) = -2P_2(t) - (1 + P_2(t)) \times \left(P_1(t) + P_2(t) + \frac{1}{2}P_2(t)^2 + \int_{t-D_1}^{t} P_2(\theta)d\theta\right), \quad (68)$$

where the predictors are given explicitly in terms of X_t and U_t as

$$P_1(t) = X_1(t) + \int_{t-D_2}^{t} \left(P_2(\theta - D_1) + P_2(\theta)^2\right) d\theta \quad (69)$$

$$P_2(t) = X_2(t) + \int_{t-D_2}^{t} U(\theta)d\theta, \quad (70)$$

with initial conditions

$$P_1(\theta) = X_1(0) + \int_{-D_2}^{\theta} \left(P_2(s - D_1) + P_2(s)^2\right) ds, \quad \text{for all } -D_2 \le \theta \le 0 \quad (71)$$

$$P_2(\theta) = X_2(0) + \int_{-D_2}^{\theta} U(s)ds, \quad \text{for all } -D_2 \le \theta \le 0 \quad (72)$$

$$P_2(s) = X_2(s + D_2), \quad \text{for all } -D_1 - D_2 \le s \le -D_2. \quad (73)$$

We choose $D_1 = 1$ and $D_2 = 2$. The initial conditions for the system and the actuator are chosen as $X_1(0) = 1$, $X_2(s) = 0.5$, for all $-1 \le s \le 0$, and $U(\theta) = 0$, for all $-2 \le \theta \le 0$, respectively. In Fig. 1 we show the response of the system (65)–(66) with the predictor feedback law (68)–(73) (solid) and the nominal, uncompensated controller (67) (dashed). In Fig. 2 we show the control effort. One can observe that the predictor feedback law globally asymptotically stabilizes system (65)–(66), whereas the closed-loop system, under the nominal, uncompensated control design is unstable.

Example 2. Consider the following scalar, linear system [38]

$$\dot{X}(t) = X(t-1) + U(t - D_2), \quad (74)$$

where $D_2 > 0$ can be arbitrary long. System (74) is a special case of system (1) with $f(X_t, U(t - D_2)) = X(t-1) + U(t - D_2)$. A nominal control law that stabilizes system (74) when $D_2 = 0$ is

$$U(t) = -X(t-1) - X(t). \quad (75)$$

For $D_2 > 0$ the predictor feedback law is

$$U(t) = -P(t-1) - P(t), \quad (76)$$

where

$$P(t) = X(t) + \int_{t-D_2}^{t} (P(s-1) + U(s))\, ds, \quad (77)$$

with initial condition

$$P(\theta) = X(0) + \int_{-D_2}^{\theta} (P(s-1) + U(s))\, ds, \quad \text{for all } -D_2 \le \theta \le 0 \quad (78)$$

$$P(s) = X(s + D_2), \quad \text{for all } -1 - D_2 \le s \le -D_2. \quad (79)$$

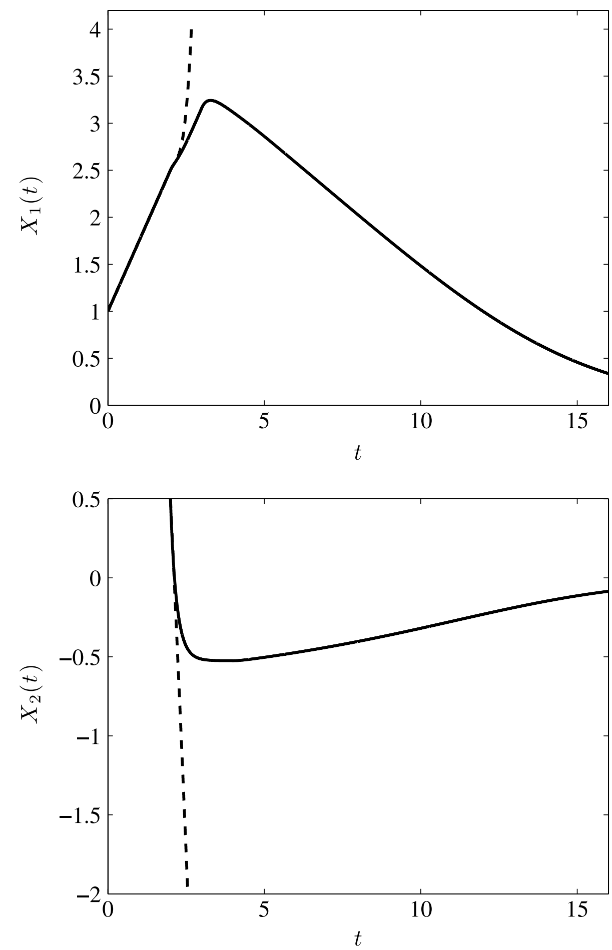

Fig. 1. Solid line: The response of the system (65)–(66) with the predictor feedback law (68)–(73) for initial conditions $X_1(0) = 1$, $X_2(s) = 0.5$, for all $-1 \le s \le 0$, and $U(\theta) = 0$, for all $-2 \le \theta \le 0$. Dashed line: The response of the system (65)–(66) with the nominal feedback law (67) for the same initial conditions.

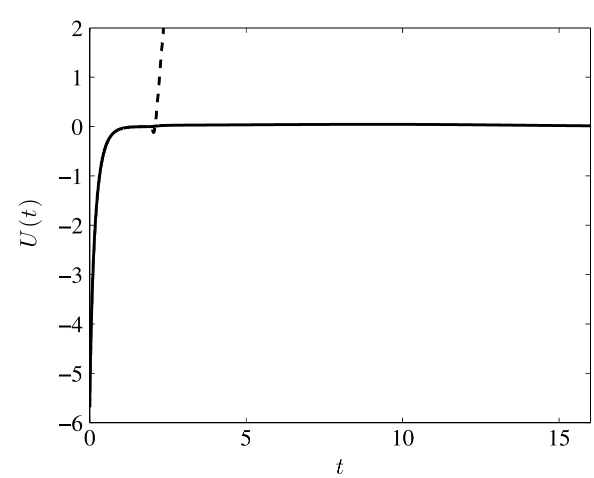

Fig. 2. Solid line: The control effort (68) for initial conditions $X_1(0) = 1$, $X_2(s) = 0.5$, for all $-1 \le s \le 0$, and $U(\theta) = 0$, for all $-2 \le \theta \le 0$. Dashed line: The control effort (67) for the same initial conditions.

Using the fact that $P(s) = X(s + D_2)$, for all $s \ge -1 - D_2$ (see Section 2) one can conclude that the control law (76)–(79) is the same with the one derived in [30] as an application of the general methodology developed in [30] for the case of linear systems.

6. Conclusions

We present a predictor feedback design for nonlinear systems with simultaneous input and state delays. We prove global asymptotic stability of the closed-loop system with the aid of a Lyapunov functional that we construct, or by using estimates on closed-loop solutions. We illustrate our design with two examples.

This paper opens an opportunity to consider problems of compensation of input delay in other, more complex nonlinear infinite-dimensional systems than systems with internal delays. For example, one can consider the problem of compensation of input delay for a wave PDE with nonlinearities of superlinear growth on the uncontrolled boundary that is considered in Section IV in [39].

References

[1] G. Newell, Delays caused by a queue at a freeway exit ramp, Transp. Res. Part B 33 (1999) 337–350.
[2] M. Diagne, F. Couenne, B. Maschke, Mass transport equation with moving interface and its control as input delay system, IFAC Joint Conference, France, 2013.
[3] I. Boussaada, A. Cela, H. Mounier, S.-I Niculescu, Control of drilling vibrations: A time-delay system-based approach, IFAC Joint Conference, France, 2013.
[4] D. Bresch-Pietri, T. Leroy, J. Chauvin, N. Petit, Practical delay modeling of externally recirculated burned gas fraction for Spark-Ignited engines, IFAC Workshop on Time Delay Systems, 2013.
[5] N. Bekiaris-Liberis, M. Jankovic, M. Krstic, PDE-based analysis and control of the oxygen storage level in three-way catalytic converters, IEEE Conf. Decis. Control (2012).
[6] D. Bresch-Pietri, T. Leroy, N. Petit, Control-oriented time-varying input-delayed temperature model for SI engine exhaust catalyst, American Control Conference, 2013.
[7] J.D. Bendtsen, M. Krstic, Control of thermodynamical system with input-dependent state delays, IEEE CDC (2013).
[8] W.P.M.H. Heemels, A.R. Teel, N. van de Wouw, D. Nesic, Networked control systems with communication constraints: Tradeoffs between transmission intervals, delays and performance, IEEE Trans. Autom. Control 55 (2010) 1781–1796.
[9] M. Jankovic, Control Lyapunov-Razumikhin functions and robust stabilization of time delay systems, IEEE Trans. Autom. Control 46 (2001) 1048–1060.
[10] M. Jankovic, Cross-term forwarding for time-delay systems, IEEE Trans. Autom. Control 54 (2009) 498–511.
[11] I. Karafyllis, P. Pepe, Z.-P. Jiang, Global output stability for systems described by retarded functional differential equations: Lyapunov characterizations, Eur. J. Control 14 (2008) 516–536.
[12] F. Mazenc, M. Malisoff, Z. Lin, Further results on input-to-state stability for nonlinear systems with delayed feedbacks, Automatica 44 (2008) 2415–2421.
[13] P. Pepe, Input-to-state stabilization of stabilizable, time-delay, control-affine, nonlinear systems, IEEE Trans. Autom. Control 54 (2009) 1688–1693.
[14] P. Pepe, On Sontag's formula for the input-to-state practical stabilization of retarded control-affine systems, Syst. Control Lett. 62 (2013) 1018–1025.
[15] P. Pepe, I. Karafyllis, Z.-P. Jiang, On the LiapunovKrasovskii methodology for the ISS of systems described by coupled delay differential and difference equations, Automatica 44 (2008) 2266–2273.
[16] N. Bekiaris-Liberis, M. Krstic, Compensation of state-dependent input delay for nonlinear systems, IEEE Trans. Autom. Control 58 (2013) 275–289.
[17] N. Bekiaris-Liberis, M. Krstic, Robustness of nonlinear predictor feedback laws to time- and state-dependent delay perturbations, Automatica 49 (2013) 1576–1590.
[18] D. Bresch-Pietri, J. Chauvin, N. Petit, Prediction-based feedback control of a class of processes with input-varying delay, American Control Conference, 2012.
[19] D. Bresch-Pietri, J. Chauvin, N. Petit, Invoking Halanay inequality to conclude on closed-loop stability of processes with input-varying delay, IFAC Workshop on Time Delay Systems, 2012.
[20] I. Karafyllis, M. Krstic, Nonlinear stabilization under sampled and delayed measurements, and with inputs subject to delay and zero-order hold, IEEE Trans. Autom. Control 57 (2012) 1141–1154.
[21] I. Karafyllis, M. Krstic, Stabilization of nonlinear delay systems using approximate predictors and high-gain observers, Automatica 49 (2013) 3623–3631.
[22] M. Krstic, On compensating long actuator delays in nonlinear control, IEEE Trans. Autom. Control 53 (2008) 1684–1688.
[23] M. Krstic, Input delay compensation for forward complete and feedforward nonlinear systems, IEEE Trans. Autom. Control 55 (2010) 287–303.
[24] F. Mazenc, M. Malisoff, Asymptotic stabilization for feedforward systems with delayed feedbacks, Automatica 49 (2013) 780–787.
[25] F. Mazenc, M. Malisoff, T.N. Dinh, Robustness of nonlinear systems with respect to delay and sampling of the controls, Automatica 49 (2013) 1925–1931.
[26] F. Mazenc, S.I. Niculescu, M. Bekaik, Backstepping for nonlinear systems with delay in the input revisited, SIAM J. Control Optim. 49 (2011) 2239–2262.
[27] F. Mazenc, S.-I. Niculescu, M. Bekaik, Stabilization of time-varying nonlinear systems with distributed delay by feedback of plant's state, IEEE Trans. Autom. Control 58 (2013) 264–269.
[28] N. Bekiaris-Liberis, M. Krstic, Stabilization of linear strict-feedback systems with delayed integrators, Automatica 46 (2010) 1902–1910.
[29] M. Jankovic, Recursive predictor design for state and output feedback controllers for linear time delay systems, Automatica 46 (2010) 510–517.
[30] V.L. Kharitonov, An extension of the prediction scheme to the case of systems with both input and state delay, Automatica 50 (2014) 211–217.
[31] I. Karafyllis, Finite-time global stabilization by means of time-varying distributed delay feedback, SIAM J. Control Optim. 45 (2006) 320–342.
[32] N. Bekiaris-Liberis, M. Krstic, Compensation of time-varying input and state delays for nonlinear systems, J. Dyn. Syst. Meas. Control 134 (2012) 011009.
[33] H. Khalil, Nonlinear Systems, third ed., Prentice Hall, 2002.
[34] I. Karafyllis, The non-uniform in time small-gain theorem for a wide class of control systems with outputs, Eur. J. Control 10 (2004) 307–323.
[35] D. Angeli, E.D. Sontag, Y. Wang, A characterization of integral input to state stability, IEEE Trans. Autom. Control 45 (2000) 1082–1097.
[36] J.K. Hale, S.M. Verduyn Lunel, Introduction to Functional Differential Equations, Springer-Verlag, New York, 1993.
[37] I. Karafyllis, Z.-P. Jiang, Stability and Stabilization of Nonlinear Systems, Springer, London, 2011.
[38] M. Krstic, Delay Compensation for Nonlinear, Adaptive, and PDE Systems, Birkhauser, 2009.
[39] N. Bekiaris-Liberis, M. Krstic, Compensation of wave actuator dynamics for nonlinear systems, IEEE Trans. Autom. Control (2014) in press.