Nonlinear Local Stabilization of a Viscous Hamilton-Jacobi PDE

Nikolaos Bekiaris-Liberis and Alexandre M. Bayen

***Abstract*—We consider the boundary stabilization problem of the non-uniform equilibrium profiles of a viscous *Hamilton-Jacobi* (HJ) *Partial Differential Equation* (PDE) with parabolic concave Hamiltonian. We design a nonlinear full-state feedback control law, assuming Neumann actuation, which achieves an arbitrary rate of convergence to the equilibrium. Our design is based on a feedback linearizing transformation which is locally invertible. We prove local exponential stability of the closed-loop system in the H^1 norm, by constructing a Lyapunov functional, and provide an estimate of the region of attraction. We design an observer-based output-feedback control law, by constructing a nonlinear observer, using only boundary measurements. We illustrate the results on a benchmark example computed numerically.**

***Index Terms*— Hamilton–Jacobi (HJ), partial differential equation (PDE).**

I. Introduction

Consider the following viscous Hamilton–Jacobi PDE system:

$$u_t(x,t) = \epsilon u_{xx}(x,t) - u_x(x,t)\,(1 + u_x(x,t)) \tag{1}$$

$$u_x(0,t) = U_0(t) \tag{2}$$

$$u_x(1,t) = U_1(t) \tag{3}$$

where u is the PDE state, $x \in [0,1]$ is the spatial domain, $t \geq 0$ is time, $\epsilon > 0$ is a viscosity coefficient, and U_0, U_1 are control variables. System (1)–(3) is the viscous approximation of a macroscopic description of the dynamics of traffic flow on a highway, in which u represents the so-called Moskowitz function [2], [11]. The value of the Moskowitz function $M = u(x,t)$ is interpreted as the "label" of a given vehicle at x and t, along a road segment [29]. Viscous Hamilton-Jacobi PDEs also appear in optimal control of stochastic systems [4], whereas they belong to the class of nonlinear parabolic PDEs which appear in applications such as, for example, plasma systems [7], fluids [9], and chemical reactors [10]. The inviscid version of system (1)–(3) is a Hamilton-Jacobi PDE which originates from a first-order hyperbolic PDE describing a conservation law for the traffic density [5], with a parabolic concave flux function (a.k.a. *Greenshields* flux function [15]) given by $F(p) = p(1-p)$ (which is quadratic in p similarly to the inviscid Burgers equation), and is obtained after applying a change of variables on the density [11]. The Greenshields flux function becomes a Greenshields Hamiltonian in the Hamilton-Jacobi formulation. Control and estimation of the inviscid version of system (1)–(3) is a different problem which we do not consider in this article, but it is investigated in [11]–[13], [23], and in [1], [6] in conservation law form.

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In the present article we consider the problem of nonlinear boundary control of the viscous Hamilton-Jacobi PDE (1)–(3), which can be viewed as the Hamilton-Jacobi counterpart of the boundary control problem of the viscous Burgers PDE, for which explicit design approaches exist in the literature [3], [8], [9], [18], [20], [21], [28]. Results dealing with the nonlinear boundary stabilization of more general classes of nonlinear parabolic PDE systems also exist [7], [22], [24], [25], [31]–[33]. In particular, the control design methodologies in [20], [21], [32], [33] are inspired from techniques originally developed for finite-dimensional nonlinear systems, namely, feedback linearization [17] and backstepping [19].

We design a nonlinear full-state feedback control law for the boundary stabilization of the non-uniform stationary profiles of the viscous Hamilton-Jacobi PDE with Greenshields Hamiltonian and Neumann actuation (1)–(3), which are not asymptotically stable in open-loop (Section III). Our design is based on a linearizing change of variables, inspired from the Hopf-Cole transformation [14], [16], which, together with the choice of the control laws, transform the system to a linear diffusion-advection system (see also [20], [21] for the design of feedback linearizing control laws in the case of the viscous Burgers equation). We stabilize the linearized system using backstepping [26], achieving an arbitrary decay rate. We prove local exponential stability of the closed-loop nonlinear system in the H^1 norm, by constructing a Lyapunov functional, with the aid of which we provide an estimate of the region of attraction (Section IV). We also design a nonlinear observer, using only boundary measurements, which we employ in an output-feedback controller (Section V). We prove that the observer-based output-feedback controller achieves local exponential stabilization (in the H^1 norm of the PDE and observer states) of the equilibrium profiles, and give an estimate of the region of attraction. A nonlinear collocated static output-feedback control design, as in the case with nonlinear "radiation boundary conditions" (see [3], [18] for the case of the viscous Burgers equation), is also possible, yet, without achieving an arbitrary decay rate of the closed-loop solutions (Section III). Finally, we illustrate our full-state feedback controller with a numerical example (Section VI).

Notation: We use the common definition of class $\mathcal{K}$, $\mathcal{K}_\infty$, and $\mathcal{KL}$ functions from [17]. For a function $u \in L^2(0,1)$ we denote by $\|u(t)\|_{L^2}$ the norm $\|u(t)\|_{L^2} = \sqrt{\int_0^1 u(x,t)^2\,dx}$. For $u \in H^1(0,1)$ we denote by $\|u(t)\|_{H^1}$ the norm $\|u(t)\|_{H^1} = \sqrt{\int_0^1 u(x,t)^2\,dx} + \sqrt{\int_0^1 u_x(x,t)^2\,dx}$. For $u \in H^2(0,1)$ we denote by $\|u\|_{H^2}$ the norm $\|u(t)\|_{H^2} = \sqrt{\int_0^1 u(x,t)^2\,dx} + \sqrt{\int_0^1 u_x(x,t)^2\,dx} + \sqrt{\int_0^1 u_{xx}(x,t)^2\,dx}$. Norms in time and space are given by $\|u\|_{H_T^{2,0}} = \sqrt{\int_0^T \|u(t)\|_{H^2}^2\,dt}$, $\|u\|_{H_T^{2,1}} = \|u\|_{H_T^{2,0}} + \|u_t\|_{H_T^{2,0}}$, and we denote $H^{2,0} = H_\infty^{2,0}$, $H^{2,1} = H_\infty^{2,1}$. We denote by $C_T^{2,1}((0,1)\times(0,T))$ the space of functions that have continuous spatial derivatives of order 2 and continuous time derivatives of order 1 on $(0,1)\times(0,T)$, and define $C_\infty^{2,1} = C^{2,1}$. We denote by $C^j(A)$ the space of functions that have continuous derivatives of order j on A. We denote an initial condition as $u_0(x) = u(x,0)$, for all $x \in [0,1]$.

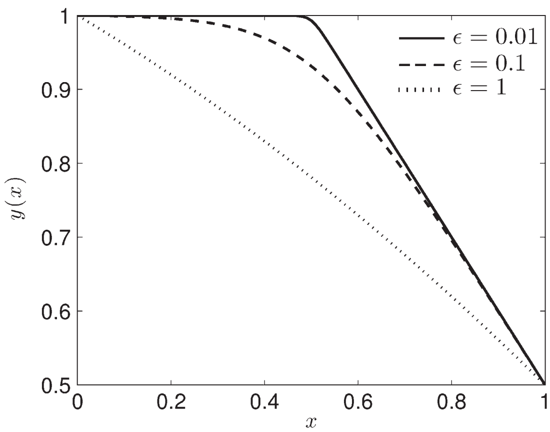

Fig. 1. The equilibrium profile (6) for $y(0) = 1$, $\sigma = 1$, and three different values of the viscosity coefficient ϵ. Solid: $\epsilon = 0.01$. Dashed: $\epsilon = 0.1$. Dotted: $\epsilon = 1$. As ϵ goes to zero, the equilibrium profile becomes non-differentiable.

II. Equilibrium Profiles and Their Open-Loop Stability Properties

The equilibrium y of system (1)–(3) satisfies the ODE in x

$$\epsilon y(x)'' - y'(x)\left(1 + y'(x)\right) = 0 \tag{4}$$

which gives

$$y'(x) = -\frac{1}{1 + c^* e^{-\frac{x}{\epsilon}}} \tag{5}$$

where $c^* \in \mathbb{R}$ is arbitrary. We stabilize the equilibrium profile (5) for any $y(0) \in \mathbb{R}$ and for c^* such that $-1 < c^*$ or $c^* < -e^{1/\epsilon}$, which guarantees that y' is continuous $\forall x \in [0,1]$ [and hence, so is y'' according to (4)]. Setting $c^* = \sigma e^{1/2\epsilon}$ we get

$$y(x) = y(0) - x - \epsilon \log\left(\frac{1 + \sigma e^{-\frac{x-\frac{1}{2}}{\epsilon}}}{1 + \sigma e^{\frac{1}{2\epsilon}}}\right). \tag{6}$$

Although c^* could be negative, the choice $c^* \geq 0$ in (5) has its own interest. Relation (5) for $c^* \geq 0$ guarantees that $-1 \leq y'(x) \leq 0$, $\forall x \in [0,1]$. This implies that the traffic density at equilibrium, which is equal to minus the spatial derivative of the Moskowitz function u [11], [29], is bounded below by zero and above by one. This is consistent with traffic models, in which, the density varies on the interval between the roots of the Hamiltonian which in the present case are 0 and -1 [11].

The equilibrium profile (6) for $y(0) = 1$, $\sigma = 1$, and its derivative, for several values of ϵ are shown in Fig. 1. As $\epsilon \to 0$, the equilibrium profile of u becomes non-differentiable, with the singularity located at $x = 1/2$ (one could change the location of this singularity by choosing a different c^*). This non-differentiable profile is the equilibrium profile of the inviscid version of (1)–(3) which we do not consider here.

We shift the equilibrium of system (1)–(3) to the origin. Defining $\tilde{u} = u - y$ we get that $\tilde{u}$ satisfies

$$\begin{aligned}\tilde{u}_t(x,t) =& \epsilon \tilde{u}_{xx}(x,t) - \tilde{u}_x(x,t)\left(1 + \tilde{u}_x(x,t)\right)\\ &- 2y'(x)\tilde{u}_x(x,t) \end{aligned}\tag{7}$$

$$\tilde{u}_x(0,t) = \tilde{U}_0(t) \tag{8}$$

$$\tilde{u}_x(1,t) = \tilde{U}_1(t) \tag{9}$$

where

$$\tilde{U}_0(t) = U_0(t) - y'(0) \tag{10}$$

$$\tilde{U}_1(t) = U_1(t) - y'(1). \tag{11}$$

Any constant could be an equilibrium of (7)–(9). Hence, the zero solution of (7)–(9) is not asymptotically stable. Thus, a control design is needed, which asymptotically stabilizes (7)–(9) to the origin. In fact, the linearized system has a zero eigenvalue independently of $y(0)$, ϵ, and σ. Linearizing (7)–(9) around zero and defining $\zeta = \tilde{u}e^{-(1/2\epsilon)\int_0^x (1+2y'(s))\, ds}$, in order to eliminate the advection term, we get $\zeta_t(x,t) = \epsilon\zeta_{xx}(x,t) - (1/4\epsilon)\zeta(x,t)$, $\zeta_x(0,t) = r_1\zeta(0,t)$, $\zeta_x(1,t) = r_2\zeta(1,t)$, where

$$r_1 = \frac{1 - \sigma e^{\frac{1}{2\epsilon}}}{2\epsilon\left(1 + \sigma e^{\frac{1}{2\epsilon}}\right)} \tag{12}$$

$$r_2 = \frac{1 - \sigma e^{-\frac{1}{2\epsilon}}}{2\epsilon\left(1 + \sigma e^{-\frac{1}{2\epsilon}}\right)}. \tag{13}$$

The eigenvalues of the solution to the ζ system include zero with eigenfunction $\phi(x) = e^{(1/2\epsilon)x} + \sigma e^{1/2\epsilon} e^{-(1/2\epsilon)x}$.

III. Controller Design

A. Feedback Linearizing Transformation

In this section we design the controllers $\tilde{U}_0, \tilde{U}_1$ in order to asymptotically stabilize system (7)–(9). We first linearize (7)–(9). Introducing the following locally invertible transformation:

$$\tilde{v}(x,t) = e^{-\frac{1}{\epsilon}\tilde{u}(x,t)} - 1 \tag{14}$$

and choosing the control laws as

$$\tilde{U}_0(t) = -\epsilon e^{\frac{1}{\epsilon}\tilde{u}(0,t)}\tilde{V}_0(t) \tag{15}$$

$$\tilde{U}_1(t) = -\epsilon e^{\frac{1}{\epsilon}\tilde{u}(1,t)}\tilde{V}_1(t) \tag{16}$$

where $\tilde{V}_0$, $\tilde{V}_1$ are the new control variables yet to be chosen, we transform system (7)–(9) to

$$\tilde{v}_t(x,t) = \epsilon\tilde{v}_{xx}(x,t) - \left(1 + 2y'(x)\right)\tilde{v}_x(x,t) \tag{17}$$

$$\tilde{v}_x(0,t) = \tilde{V}_0(t) \tag{18}$$

$$\tilde{v}_x(1,t) = \tilde{V}_1(t). \tag{19}$$

The inspiration for (14) is the Hopf-Cole transformation [14], [16] and the fact that $h = 2u_x$ satisfies $h_t = \epsilon h_{xx} - ((h^2/2) + h)_x$. A feedback linearizing transformation for the Burgers PDE is introduced in [20].

The inverse of transformation (14) is given by

$$\tilde{u}(x,t) = -\epsilon \log\left(\tilde{v}(x,t) + 1\right) \tag{20}$$

which is well-defined when the initial conditions and solutions of the system satisfy for some $c \in (0,1]$

$$\sup_{x\in[0,1]} |\tilde{v}(x,t)| < c, \quad \text{for all } t \geq 0. \tag{21}$$

B. Full-State Feedback Controller

Our next step is to choose the control variables $\tilde{V}_0$ and $\tilde{V}_1$ in order to stabilize the linear diffusion-advection PDE (17)–(19) with an arbitrary decay rate of convergence. We first define the transformation

$$v(x,t) = \tilde{v}(x,t)e^{-\frac{1}{2\epsilon}\int_0^x\left(1+2y'(s)\right)ds} \tag{22}$$

in order to eliminate the advection term in (17), and we choose the control variables $\tilde{V}_0$, $\tilde{V}_1$ as

$$\tilde{V}_0(t) = -r_1\tilde{v}(0,t) \tag{23}$$

$$\tilde{V}_1(t) = e^{\frac{1}{2\epsilon}\int_0^1\left(1+2y'(x)\right)dx}V_1(t) - r_2\tilde{v}(1,t) \tag{24}$$

where r_1, r_2 are given in (12), (13), and V_1 is a new control variable yet to be designed, in order to get

$$v_t(x,t) = \epsilon v_{xx}(x,t) - \frac{1}{4\epsilon}v(x,t) \tag{25}$$

$$v_x(0,t) = 0 \tag{26}$$

$$v_x(1,t) = V_1(t). \tag{27}$$

We employ next backstepping for stabilization of system (25)–(27) [26]. Using the transformation

$$w(x,t) = v(x,t) - \int_0^x k(x,y)v(y,t)\,dy \tag{28}$$

system (25)–(27) is mapped to the "target system"

$$w_t(x,t) = \epsilon w_{xx}(x,t) - \left(\frac{1}{4\epsilon}+c_1\right)w(x,t) \tag{29}$$

$$w_x(0,t) = 0 \tag{30}$$

$$w_x(1,t) = 0 \tag{31}$$

where $c_1 > 0$ is arbitrary, when the gain kernel k satisfies $k_{xx}(x,y) - k_{yy}(x,y) = (c_1/\epsilon)k(x,y)$, $(dk(x,x))/dx = -(c_1/2\epsilon)$, $k_y(x,0) = 0$, with $k(0,0) = 0$, such that (30) is satisfied given (26), and the control law V_1 is chosen as

$$V_1(t) = k(1,1)v(1,t) + \int_0^1 k_x(1,y)v(y,t)\,dy. \tag{32}$$

In [26] it is shown that $k \in C^2(E)$, where $E = \{(x,y) : 0 \le y \le x \le 1\}$, and that k is given by

$$k(x,y) = -\frac{c_1}{\epsilon}x\frac{I_1\left(\sqrt{\frac{c_1}{\epsilon}(x^2-y^2)}\right)}{\sqrt{\frac{c_1}{\epsilon}(x^2-y^2)}} \tag{33}$$

where I_1 is a modified Bessel function of order one. Combining relations (14), (22)–(24), (32), the control laws (15), (16) are expressed in terms of the original variable $\tilde{u}$

$$\tilde{U}_0(t) = -\epsilon r_1\left(e^{\frac{1}{\epsilon}\tilde{u}(0,t)}-1\right) \tag{34}$$

$$\tilde{U}_1(t) = \epsilon\left(-r_2+k(1,1)\right)\left(e^{\frac{1}{\epsilon}\tilde{u}(1,t)}-1\right) + \frac{\epsilon e^{\frac{1}{\epsilon}\tilde{u}(1,t)}}{1+\sigma e^{-\frac{1}{2\epsilon}}} \times\int_0^1 k_x(1,y)\left(e^{\frac{y-1}{2\epsilon}}+\sigma e^{-\frac{y}{2\epsilon}}\right) \times\left(1-e^{-\frac{1}{\epsilon}\tilde{u}(y,t)}\right)dy \tag{35}$$

where r_1, r_2 are given in (12), (13). The inverse of (28) is

$$v(x,t) = w(x,t) + \int_0^x l(x,y)w(y,t)\,dy \tag{36}$$

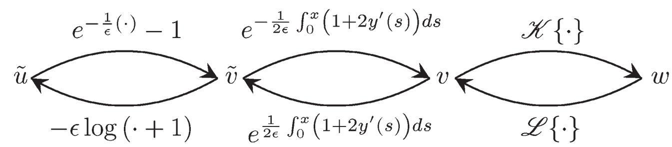

Fig. 2. The interconnections between $\tilde{u}$, $\tilde{v}$, v, and w involved in transformations (14), (20), (22), (28), and (36). The operators $\mathcal{K}\{\cdot\}$ and $\mathcal{L}\{\cdot\}$ are defined as $\mathcal{K}\{v\}(x) = v(x) - \int_0^x k(x,y)v(y)\,dy$ and $\mathcal{L}\{w\}(x) = w(x) + \int_0^x l(x,y)w(y)\,dy$ respectively. Transformation (14) is locally invertible.

which is well-defined, namely, $l \in C^2(E)$ [26]. Fig. 2 shows the interconnections between $\tilde{u}$, $\tilde{v}$, v, w.

System (1)–(3) can be stabilized using only one feedback controller. Consider the case of using a feedback controller at $x = 1$ (analogously for $x = 0$). Setting $\tilde{U}_0(t) = 0$ in (8), i.e., setting $\tilde{V}_0(t) = 0$ in (19) we get that v satisfies (25), (27) together with $v_x(0,t) = r_1 v(0,t)$. System (7)–(9) is stabilized by choosing a feedback controller $\tilde{U}_1$ since (25), (27) with $v_x(0,t) = r_1 v(0,t)$ is stabilized for any $r_1 \in \mathbb{R}$ by designing V_1 via backstepping (Theorem 4 in [26]). Yet, in this case, the controller's gain c_1 is not arbitrary but it is chosen large enough since r_1 in (15) can be negative (Theorem 4 in [26]).

C. Static Collocated Output-Feedback Controller

Setting $c_1 = 0$ in (29) we get $k = 0$, and hence, we obtain the new simpler controllers $\tilde{U}_0(t) = -\epsilon r_1(e^{(1/\epsilon)\tilde{u}(0,t)} - 1)$, $\tilde{U}_1(t) = -\epsilon r_2(e^{(1/\epsilon)\tilde{u}(1,t)} - 1)$. In this case v satisfies (25)–(27) with $V_1 = 0$ which is exponentially stable but not with an arbitrary decay rate. Yet, the new controllers are given only in terms of $\tilde{u}(0,t)$, $\tilde{u}(1,t)$. This enables one to design a static output-feedback controller, in contrast to (35) which requires measurement of the full state $\tilde{u}(x,t)$, $\forall x$.

IV. Stability Analysis

Theorem 1: Let $c_1 \ge 0$ be arbitrary. Consider system (7)–(9) together with the control laws (34), (35), (33). There exists a positive constant μ_1 (which depends on c_1) such that for all initial conditions $\tilde{u}_0 \in H^2(0,1)$ which are compatible with the feedback laws (34), (35) and which satisfy

$$\|\tilde{u}(0)\|_{H^1} < \alpha_1^{-1}\left(\frac{c}{2\mu_1}\right) \tag{37}$$

$$\alpha_1(s) = \frac{3s}{\epsilon}e^{\frac{2s}{\epsilon}} \tag{38}$$

for some $0 < c < 1$, the following holds:

$$\|\tilde{u}(t)\|_{H^1} \le \alpha\left(\|\tilde{u}(0)\|_{H^1}\right)e^{-\left(c_1+\frac{1}{4\epsilon}\right)t}, \quad \forall t \ge 0 \tag{39}$$

$$\alpha(s) = \frac{3\mu_1}{1-c}e^{\frac{2s}{\epsilon}}s. \tag{40}$$

Moreover, the closed-loop system has a unique solution $\tilde{u} \in H^{2,1}((0,1)\times(0,\infty))$.

Note that μ_1 is given explicitly as $\mu_1 = MM_1M_2$, where $M = 24\sqrt{2}e^{2m}(1+\sqrt{2}(1+m))^2$, $m = 1 + 2\sup_{x\in[0,1]}|y'(x)|/2\epsilon$, $M_1 = 1 + \sup_{(x,y)\in E}|l(x,y)| + \sup_{(x,y)\in E}|l_x(x,y)|$, and $M_2 = 1 + \sup_{(x,y)\in E}|k(x,y)| + \sup_{(x,y)\in E}|k_x(x,y)|$. The compatibility condition can be written explicitly as $\tilde{u}_{0x}(0) = -\epsilon r_1(e^{(1/\epsilon)\tilde{u}_0(0)} - 1)$, $\tilde{u}_{0x}(1) = \epsilon(-r_2 + k(1,1))(e^{(1/\epsilon)\tilde{u}_0(1)} - 1) + (\epsilon e^{(1/\epsilon)\tilde{u}_0(1)}/1 + \sigma e^{-(1/2\epsilon)})\int_0^1 k_x(1,y)(e^{(y-1)/2\epsilon} + \sigma e^{-(y/2\epsilon)}) \times (1 - e^{-(1/\epsilon)\tilde{u}_0(y)})\,dy$.

The proof of Theorem 1 is based on the next three lemmas whose proofs are provided in Appendix A.

Lemma 1: If $\tilde{u} \in H^1(0,1)$ then $\tilde{v} \in H^1(0,1)$ and the following holds:

$$\|\tilde{v}(t)\|_{H^1} \leq \alpha_1 \left(\|\tilde{u}(t)\|_{H^1}\right) \tag{41}$$

where the class $\mathcal{K}_\infty$ function α_1 is defined in (38). Moreover, if $\tilde{u} \in H^2(0,1)$ then $\tilde{v} \in H^2(0,1)$.

Lemma 2: For all solutions of the system that satisfy (21) for some $0 < c < 1$, if $\tilde{v} \in H^1(0,1)$ then $\tilde{u} \in H^1(0,1)$ and the following holds:

$$\|\tilde{u}(t)\|_{H^1} \leq \frac{\epsilon}{1-c} \|\tilde{v}(t)\|_{H^1}. \tag{42}$$

Moreover, for all solutions satisfying (21) for some $0 < c < 1$, if $\tilde{v} \in H^2(0,1)$ then $\tilde{u} \in H^2(0,1)$.

Lemma 3: Let $c_1 \geq 0$ be arbitrary and μ_1 be as in Theorem 1. The following holds:

$$\|\tilde{v}(t)\|_{H^1} \leq \mu_1 \|\tilde{v}(0)\|_{H^1} e^{-\left(c_1+\frac{1}{4\epsilon}\right)t}, \quad \text{for all } t \geq 0. \tag{43}$$

Proof of Theorem 1: Combining (41) and (43) we get

$$\|\tilde{v}(t)\|_{H^1} \leq \mu_1\alpha_1 \left(\|\tilde{u}(0)\|_{H^1}\right) e^{-\left(c_1+\frac{1}{4\epsilon}\right)t}. \tag{44}$$

With (37) we get $\|\tilde{v}(t)\|_{H^1} < (c/2)$. Hence, since $\sup_{x\in[0,1]} |\tilde{v}(x,t)| \leq 2\|\tilde{v}(t)\|_{H^1}$, for any $\tilde{v} \in H^1(0,1)$, we conclude that relation (21) is satisfied. Thus, using (42), (38) we arrive at (39), (40). Existence and uniqueness of a solution $\tilde{u} \in H^{2,1}((0,1) \times (0,\infty))$ follows from the target system (29)–(31), transformations (28), (36) and Lemmas 1, 2, by using almost identical arguments to [20] (Section VII). ■

V. Observer-Based Controller Design

A. Observer Design

For assigning an arbitrarily fast decay rate to the closed-loop system the control law $\tilde{U}_1$ is chosen according to (35), which requires measurement of the full state $\tilde{u}(x,t)$, $x \in [0,1]$. Yet, when only boundary measurements are available (rather than the full state $\tilde{u}(x,t)$, $x \in [0,1]$), one has to combine the control design with an observer which employs measurements only of $\tilde{u}(0,t)$ and $\tilde{u}(1,t)$. Since $\tilde{U}_0$ in (34) depends only on $\tilde{u}(0,t)$, we design next V_1 in order to stabilize (25)–(27) (with a prescribed decay rate c_1) using only a measurement of $\tilde{u}(1,t)$. We employ an observer which is a copy of (25)–(27) plus output injection

$$\hat{v}_t(x,t) = \epsilon\hat{v}_{xx}(x,t) - \frac{1}{4\epsilon}\hat{v}(x,t) + p_1(x)\left(\left(e^{-\frac{1}{\epsilon}\tilde{u}(1,t)} - 1\right)e^{-\frac{1}{2\epsilon}\int_0^1\left(1+2y'(s)\right)ds} - \hat{v}(1,t)\right) \tag{45}$$

$$\hat{v}_x(0,t) = 0 \tag{46}$$

$$\hat{v}_x(1,t) = V_1(t) - p_{10}\left(\left(e^{-\frac{1}{\epsilon}\tilde{u}(1,t)} - 1\right) \times e^{-\frac{1}{2\epsilon}\int_0^1\left(1+2y'(s)\right)ds} - \hat{v}(1,t)\right) \tag{47}$$

where y is given in (6) and $p_1(x)$, p_{10} are yet to be designed. Note that

$$v(1,t) = \left(e^{-\frac{1}{\epsilon}\tilde{u}(1,t)} - 1\right) e^{-\frac{1}{2\epsilon}\int_0^1\left(1+2y'(s)\right)ds} \tag{48}$$

which is measured. Hence, $e = v - \hat{v}$ satisfies $e_t(x,t) = \epsilon e_{xx}(x,t) - (1/4\epsilon)e(x,t) - p_1(x)e(1,t)$, $e_x(0,t) = 0$, $e_x(1,t) = p_{10}e(1,t)$. We design p_1, p_{10} via backstepping [27]. Employing the invertible transformation $e(x,t) = z(x,t) - \int_x^1 p(x,y)z(y,t)\,dy$, we map e system to $z_t(x,t) = \epsilon z_{xx}(x,t) - ((1/4\epsilon) + c_2)z(x,t)$, $z_x(0,t) = z_x(1,t) = 0$, if p_1, p_{10} satisfy

$$p_1(x) = -\epsilon p_y(x,1) \tag{49}$$

$$p_{10} = p(1,1) \tag{50}$$

where p satisfies $p_{xx}(x,y) - p_{yy}(x,y) = -(c_2/\epsilon)p(x,y)$, $(dp(x,x))/dx = -(c_2/2\epsilon)$, $p_x(0,y) = 0$, with $p(0,0) = 0$, such that $z_x(0,t) = 0$ holds given $e_x(0,t) = 0$. Existence and uniqueness of $p \in C^2(B)$, where $B = \{(x,y) : 0 \leq x \leq y \leq 1\}$, follows [27]. In fact, $p(x,y) = -(c_2/\epsilon)y(I_1(\sqrt{c_2/\epsilon(y^2-x^2)})/\sqrt{(c_2/\epsilon)(y^2-x^2)})$ [27].

B. Output-Feedback Controller and Stability Analysis

Combining (32) with (14) and (22), the controller V_1 is

$$V_1(t) = k(1,1)e^{-\frac{1}{2\epsilon}\int_0^1\left(1+2y'(s)\right)ds}\left(e^{-\frac{1}{\epsilon}\tilde{u}(1,t)} - 1\right) + \int_0^1 k_x(1,y)\hat{v}(y,t)\,dy \tag{51}$$

and hence, using (24) we get from (16), (14) that

$$\tilde{U}_1(t) = \epsilon\left(-r_2 + k(1,1)\right)\left(e^{\frac{1}{\epsilon}\tilde{u}(1,t)} - 1\right) - \epsilon e^{\frac{1}{\epsilon}\tilde{u}(1,t)} \times e^{\frac{1}{2\epsilon}\int_0^1\left(1+2y'(s)\right)ds}\int_0^1 k_x(1,y)\hat{v}(y,t)\,dy \tag{52}$$

where y and r_2 are defined in (6) and (13) respectively.

Theorem 2: Consider a closed-loop system consisting of system (7)–(9), the control laws (34), (52), and the observer (45)–(47) with (49)–(51). There exist positive constants R^*, λ^*, and ν such that for all initial conditions $(\tilde{u}_0, \hat{v}_0) \in H^1(0,1) \times H^1(0,1)$ which satisfy the compatibility conditions $\tilde{u}_{0x}(0) = -\epsilon r_1(e^{(1/\epsilon)\tilde{u}_0(0)} - 1)$, $\tilde{u}_{0x}(1) = \epsilon(-r_2 + k(1,1))(e^{(1/\epsilon)\tilde{u}_0(1)} - 1) - \epsilon e^{(1/\epsilon)\tilde{u}_0(1)} e^{(1/2\epsilon)\int_0^1(1+2y'(s))ds} \int_0^1 k_x(1,y)\hat{v}_0(y)\,dy$, $\hat{v}_{0x}(0) = 0$, $\hat{v}_{0x}(1) = -p_{10}((e^{-(1/\epsilon)\tilde{u}_0(1)} - 1)e^{-(1/2\epsilon)\int_0^1(1+2y'(s))ds} - \hat{v}_0(1)) + k(1,1)e^{-(1/2\epsilon)\int_0^1(1+2y'(s))ds}(e^{-(1/\epsilon)\tilde{u}_0(1)} - 1) + \int_0^1 k_x(1,y)\hat{v}_0(y)\,dy$, and are such that

$$\Xi(0) < R^* \tag{53}$$

$$\Xi(t) = \|\tilde{u}(t)\|_{H^1} + \|\hat{v}(t)\|_{H^1} \tag{54}$$

the following holds:

$$\Xi(t) \leq \alpha^*\left(\Xi(0)\right)e^{-\lambda^* t}, \quad \text{for all } t \geq 0 \tag{55}$$

$$\alpha^*(s) = \nu\left(1 + \frac{\epsilon}{1-c}\right)\left(1 + \frac{3}{\epsilon}e^{\frac{2s}{\epsilon}}\right)s \tag{56}$$

with $0 < c < 1$. Moreover, there exists a unique solution $(\tilde{u}, \hat{v}) \in C^{2,1} \times C^{2,1}$.

Proof: The $(v, \hat{v})$ system (25)–(27), (45)–(47), together with (51) is exponentially stable in H^1. This follows from (48) and the fact that $v(x,0) \in H^1$ [which follows from Lemma 1 and (22)], that allow one to use Theorem 5 in [27] for system (25)–(27) with the observer

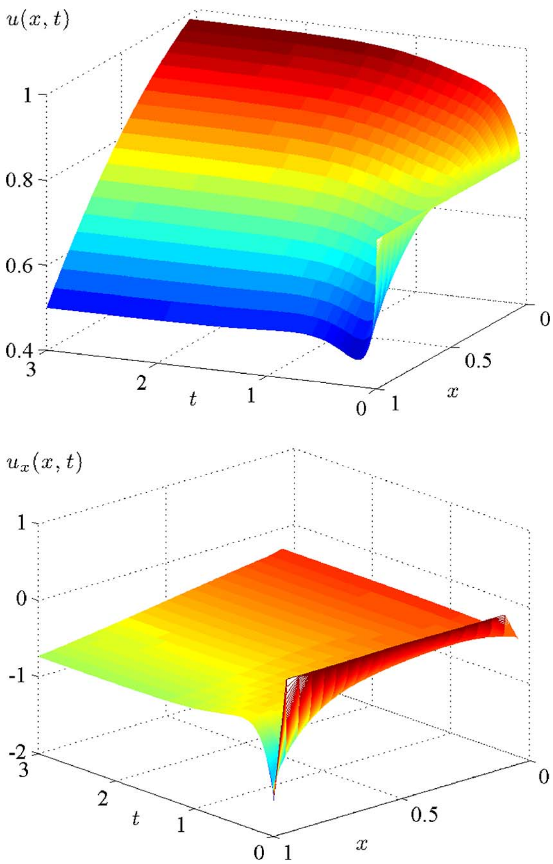

Fig. 3. The response of system (1)–(3) under the observer-based controller (34), (52) for initial conditions $u(x, 0) = 0.75$, $\hat{v}(x, 0) = 0$, $\forall\, x \in [0, 1]$.

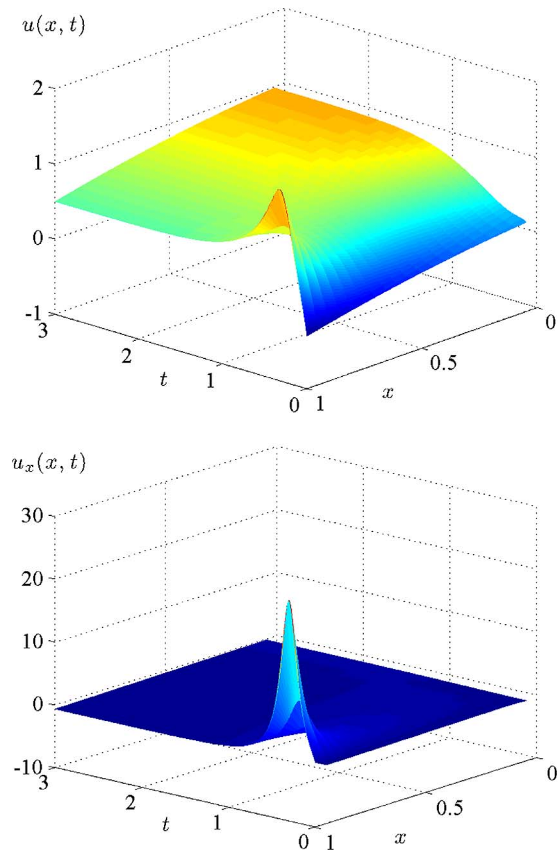

Fig. 4. The response of system (1)–(3) under the observer-based controller (34), (52) for initial condition $u(x, 0) = y(x) - 0.8 = 0.2 - x - .5\log(1 + e^{1-2x}/1 + e)$ and $\hat{v}(x, 0) = 0$, for all $x \in [0, 1]$.

(45)–(47). Hence, there exist positive constants μ^*, λ^* such that $\|\hat{v}(t)\|_{H^1} + \|v(t)\|_{H^1} \le \mu^*(\|\hat{v}(0)\|_{H^1} + \|v(0)\|_{H^1})e^{-\lambda^* t}$, $\forall t \ge 0$. Using (22) we get

$$\|v(t)\|_{H^1} \le e^m \left(1 + \sqrt{2}(1+m)\right) \|\tilde{v}(t)\|_{H^1} \quad (57)$$
$$\|\tilde{v}(t)\|_{H^1} \le e^m \left(1 + \sqrt{2}(1+m)\right) \|v(t)\|_{H^1} \quad (58)$$

where m is as in Theorem 1. Hence, $\|\hat{v}(t)\|_{H^1} + \|\tilde{v}(t)\|_{H^1} \le \nu(\|\hat{v}(0)\|_{H^1} + \|\tilde{v}(0)\|_{H^1})e^{-\lambda^* t}$, for some $\nu > 0$. From Lemma 1 [relation (41)] we conclude that $\|\hat{v}(t)\|_{H^1} + \|\tilde{v}(t)\|_{H^1} \le \rho(\|\hat{v}(0)\|_{H^1} + \|\tilde{u}(0)\|_{H^1})e^{-\lambda^* t}$ where $\rho \in \mathcal{K}_\infty$ is $\rho(s) = \nu s + \nu\alpha_1(s)$. Since $\sup_{x\in[0,1]} |\tilde{v}(x,t)| \le 2\|\tilde{v}(t)\|_{H^1}$, for any $\tilde{v} \in H^1(0,1)$, choosing $R^* = \rho^{-1}(c/2)$ we get that (21) holds. Thus, using Lemma 2 (relation (42)) we get (55), (56). Existence and uniqueness of a solution $(\tilde{u}, \hat{v}) \in C^{2,1} \times C^{2,1}$ follows from (20)–(22), by using Theorem 5 in [27]. ■

VI. Simulations

Figs. 3 and 4 show the response of (1)–(3) with $\epsilon = 0.5$, under the observer-based controller (34), (52) with $c_1 = 2$, $c_2 = 4$, that stabilizes the equilibrium profile (6) with $y(0) = 1$, $\sigma = 1$.

VII. Conclusion

From the expression of μ_1 in Theorem 1 and (33) one can observe that μ_1 increases with c_1. Hence, (37) reveals a tradeoff between the achievable region of attraction and the decay rate of the closed-loop system. Also, from (39) it is evident that a large decay rate of the closed-loop system implies a large overshoot. The same tools can be applied to the stabilization of the viscous HJ PDE given by $u_t = \epsilon u_{xx} - a u_x(b + u_x)$.

Appendix A

Proof of Lemma 1: For $f(r) = e^{-(1/\epsilon)r} - 1$ it holds $|f(r)| \le (|r|/\epsilon)e^{|r|/\epsilon}, \forall r \in \mathbb{R}$. Hence, with (14) we get $|\tilde{v}(x,t)| \le \hat{\alpha}(|\tilde{u}(x,t)|)$, $\hat{\alpha}(s) = (s/\epsilon)e^{s/\epsilon} \in \mathcal{K}_\infty$. Hence, $|\tilde{v}(x,t)| \le \hat{\alpha}(\sup_{x\in[0,1]} |\tilde{u}(x,t)|)$, $\forall x \in [0,1]$. For any $u \in H^1(0,1)$ we have $u(x,t) = u(0,t) + \int_0^x u_y(y,t)\, dy$. Hence, using Cauchy–Schwartz's inequality we get

$$|u(x,t)| \le |u(0,t)| + \sqrt{\int_0^1 u_x(x,t)^2\, dx}, \quad x \in [0,1]. \quad (A.1)$$

Since $u(0,t) = u(x,t) - \int_0^x u_y(y,t)\, dy$ we get $|u(0,t)| \le |u(x,t)| + \sqrt{\int_0^1 u_x(x,t)^2\, dx}$. Hence, by integrating we get

$$|u(0,t)| \le \|u(t)\|_{H^1}\,. \quad (A.2)$$

Since $\forall x$, $|\tilde{v}(x,t)| \le \hat{\alpha}(\sup_{x\in[0,1]} |\tilde{u}(x,t)|)$ and by (A.1), (A.2)

$$\sup_{x\in[0,1]} |\tilde{u}(x,t)| \le 2\, \|\tilde{u}(t)\|_{H^1} \quad (A.3)$$

we get $\sup_{x\in[0,1]} |\tilde{v}(x,t)| \le \hat{\alpha}(2\|\tilde{u}(t)\|_{H^1})$. Thus

$$\|\tilde{v}(t)\|_{L^2} \le \hat{\alpha}\left(2\, \|\tilde{u}(t)\|_{H^1}\right). \quad (A.4)$$

Differentiating (14) we get with (A.3), $\sqrt{\int_0^1 \tilde{v}_x(x,t)^2\, dx} \le (1/\epsilon)e^{(2/\epsilon)\|\tilde{u}(t)\|_{H^1}} \sqrt{\int_0^1 \tilde{u}_x(x,t)^2\, dx}$, and hence, with (A.4) we get $\|\tilde{v}(t)\|_{H^1} \le \hat{\alpha}(2\|\tilde{u}(t)\|_{H^1}) + (1/\epsilon)e^{(2/\epsilon)\|\tilde{u}(t)\|_{H^1}} \|\tilde{u}(t)\|_{H^1}$, which gives (41), (38). Similarly, using the fact that $\tilde{v}_{xx}(x,t) = -(1/\epsilon)\tilde{u}_{xx}$

$(x,t)e^{-(1/\epsilon)\tilde{u}(x,t)}$ $+(1/\epsilon^2)\tilde{u}_x(x,t)^2 e^{-(1/\epsilon)\tilde{u}(x,t)}$, and (A.1), (A.2) for $u=\tilde{u}_x \in H^1(0,1)$, we get that if $\tilde{u} \in H^2(0,1)$ then $\|\tilde{v}(t)\|_{H^2} \leq \alpha_2(\|\tilde{u}(t)\|_{H^2})$, $\alpha_2(s)=(\sqrt{2}/\epsilon)e^{(2/\epsilon)s}((2/\epsilon)s+1)s \in \mathcal{K}_\infty$.

Proof of Lemma 2: Using (20), (21) we get $|\tilde{u}(x,t)| \leq (\epsilon/1-c)|\tilde{v}(x,t)|$, since $|\log(r+1)| \leq (1/1-c)|r|$, $\forall |r|<c$, $0<c<1$. Hence, $\|\tilde{u}(t)\|_{L^2} \leq (\epsilon/1-c)\|\tilde{v}(t)\|_{L^2}$. Using this relation and (21) we get (42) by differentiating (20) with respect to x. From (21), $\tilde{u}_{xx}(x,t)=-\epsilon(\tilde{v}_{xx}(x,t)(\tilde{v}(x,t)+1)-\tilde{v}_x(x,t)^2/(\tilde{v}(x,t)+1)^2)$, and (A.1), (A.2) for $u=\tilde{v}_x \in H^1(0,1)$ we get $\sqrt{\int_0^1 \tilde{u}_{xx}(x,t)^2\,dx} \leq (\sqrt{2}\epsilon/1-c)\sqrt{\int_0^1 \tilde{v}_{xx}(x,t)^2\,dx}+(2\sqrt{2}\epsilon/(1-c)^2)\|\tilde{v}(t)\|_{H^1}\|\tilde{v}(t)\|_{H^2}$. Thus, using this relation and (42), $\|\tilde{u}(t)\|_{L^2} \leq (\epsilon/1-c)\|\tilde{v}(t)\|_{L^2}$, we get $\|\tilde{u}(t)\|_{H^2} \leq \alpha_4(\|\tilde{v}(t)\|_{H^2})$, $\alpha_4(s)=(\sqrt{2}\epsilon/1-c)s+(2\sqrt{2}\epsilon/(1-c)^2)s^2 \in \mathcal{K}_\infty$.

Proof of Lemma 3: Taking the L^2-inner product of (29) with w and w_{xx}, we get $(d\int_0^1 w(x,t)^2\,dx)/dt=-2\epsilon\int_0^1 w_x(x,t)^2\,dx-2\gamma\int_0^1 w(x,t)^2\,dx$ and $(d\int_0^1 w_x(x,t)^2\,dx)/dt=-2\epsilon\int_0^1 w_{xx}(x,t)^2\,dx-2\gamma\int_0^1 w_x(x,t)^2\,dx$ respectively, with $\gamma=c_1+(1/4\epsilon)$, where we used integration by parts and (30), (31). Using the Lyapunov functional $V_1(t)=(1/2)\int_0^1 w(x,t)^2\,dx+(1/2)\int_0^1 w_x(x,t)^2\,dx$, we get $\dot{V}_1(t) \leq -2(c_1+(1/4\epsilon))V_1(t)$. Hence, $\|w(t)\|_{H^1} \leq \sqrt{2}e^{-(c_1+(1/4\epsilon))t}\|w(0)\|_{H^1}$. Using (28), (36), and since $k,l \in C^2(E)$, we get $\|w(t)\|_{H^1} \leq 2\sqrt{6}\times(1+\sup_{(x,y)\in E}|k(x,y)|+\sup_{(x,y)\in E}|k_x(x,y)|)\|v(t)\|_{H^1}$ and $\|v(t)\|_{H^1} \leq 2\sqrt{6}\|w(t)\|_{H^1}(1+\sup_{(x,y)\in E}|l(x,y)|+\sup_{(x,y)\in E}|l_x(x,y)|)$. With (57), (58) we get (43).

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