Automatica 70 (2016) 195–203

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Stability of predictor-based feedback for nonlinear systems with distributed input delay☆

Nikolaos Bekiaris-Liberis[a], Miroslav Krstic[b]

[a] Department of Production Engineering & Management, Technical University of Crete, Chania, 73100, Greece

[b] Department of Mechanical & Aerospace Engineering, University of California, San Diego, La Jolla, CA 92093-0411, USA

ARTICLE INFO

Article history:
Received 13 September 2015
Received in revised form
19 January 2016
Accepted 19 March 2016

Keywords:
Predictor feedback
Distributed delay
Nonlinear systems
Delay systems

ABSTRACT

We consider Ponomarev's recent predictor-based control design for nonlinear systems with distributed input delays and remove certain restrictions to the class of systems by performing the stability analysis differently. We consider nonlinear systems that are not necessarily affine in the control input and whose vector field does not necessarily satisfy a linear growth condition. Employing a nominal feedback law, not necessarily satisfying a linear growth restriction, which globally asymptotically, and not necessarily exponentially, stabilizes a nominal transformed system, we prove global asymptotic stability of the original closed-loop system, under the predictor-based version of the nominal feedback law, utilizing estimates on solutions. We then identify a class of systems that includes systems transformable to a completely delay-free equivalent for which global asymptotic stability is shown employing similar tools. For these two classes of systems, we also provide an alternative stability proof via the construction of a novel Lyapunov functional. Although in order to help the reader to better digest the details of the introduced analysis methodology we focus on nonlinear systems without distributed delay terms, we demonstrate how the developed approach can be extended to the case of systems with distributed delay terms as well.

1. Introduction

1.1. Background and motivation

In Ponomarev (in press) the following class of systems is considered

$$\dot{X}(t) = f(X(t)) + B_1(X(t))\,U(t-D) + B_0(X(t))\,U(t) + \int_{-D}^{0} B_{\rm int}(\theta, X(t))\,U(t+\theta)d\theta, \quad (1)$$

where $X \in \mathbb{R}^n$ is state, $U \in \mathbb{R}$ is control input, $D > 0$ is a delay, $t \in \mathbb{R}$ is time, $f : \mathbb{R}^n \to \mathbb{R}^n$ is vector field, and $B_0, B_1 : \mathbb{R}^n \to \mathbb{R}^n$ and $B_{\rm int} : [-D, 0] \times \mathbb{R}^n \to \mathbb{R}^n$ are input vector fields. A predictor-based control law is designed in Ponomarev (in press) for the stabilization of (1).

In this article, we consider the following system

$$\dot{X}(t) = f(X(t), U(t-D), U(t)), \quad (2)$$

under a predictor-based control law that is constructed employing the design tools introduced in Ponomarev (in press).

Numerous recent results on the predictor-based stabilization of nonlinear systems controlled only through a single input channel with delay are reported, including systems with constant (Krstic, 2009, 2010; Mazenc & Malisoff, 2014), state-dependent (Bekiaris-Liberis & Krstic, 2013a,b,c), input-dependent (Bresch-Pietri, Chauvin, & Petit, 2014), and unknown (Bresch-Pietri & Krstic, 2014) delay, systems stabilized under sampling (Karafyllis & Krstic, 2012), positive systems (Mazenc & Niculescu, 2011), as well as the introduction of approximation and implementation schemes (Karafyllis, 2011; Karafyllis & Krstic, 2013, 2014). Despite the several recent developments, the problems of stabilization and of stability analysis of nonlinear systems of the form (1) and (2) are rarely investigated (Mazenc, Niculescu, & Bekaik, 2013; Ponomarev, in press) (see also Marquez-Martinez & Moog, 2004; Xia, Marquez-Martinez, Zagalak, & Moog, 2002 that adopt an algebraic approach), although both predictor-based design techniques, including classical reduction approaches (Artstein, 1982; Manitius &

☆ The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Hiroshi Ito under the direction of Editor Andrew R. Teel.

E-mail addresses: nikos.bekiaris@gmail.com (N. Bekiaris-Liberis), krstic@ucsd.edu (M. Krstic).

http://dx.doi.org/10.1016/j.automatica.2016.04.011
0005-1098/

Olbrot, 1979; Mondie & Michiels, 2003), optimal (Ariola & Pironti, 2008; Shuai, Lihua, & Huanshui, 2008) and robust (Chen & Zheng, 2002; Yue, 2004) control methods, and nested prediction-based control laws (Zhou, 2014), as well as analysis tools (Bekiaris-Liberis & Krstic, 2011; Fridman, 2014; Li, Zhou, & Lam, 2014; Mazenc, Niculescu, & Krstic, 2012; Ponomarev, 2016) exist for the linear case.

Besides the theoretical significance of studying systems of the form (1) and (2), which lies in the fact that the classic linear predictor-based control design approach is extended to the nonlinear case, systems of the form (1) and (2) appear in various applications such as networked control systems (Goebel, Munz, & Allgower, 2010; Roesch, Roth, & Niculescu, 2005), population dynamics (Artstein, 1982), and combustion control (Xie, Fridman, & Shaked, 2001; Zheng & Frank, 2002), among several other applications (Niculescu, 2001; Richard, 2003).

1.2. Contribution

For system (2) we design a predictor-based control law following the design procedure developed in Ponomarev (in press). Specifically, we first define the transformation Z of the state X defined as

$$p(x,t) = X(t) + \int_0^x f\left(p(y,t), u(y,t), 0\right) dy, \quad x \in [0,D] \tag{3}$$

$$Z(t) = p(D,t), \tag{4}$$

where we use the following, transport PDE representation of the actuator state $U(\theta), \theta \in [t-D, t]$,

$$u_t(x,t) = u_x(x,t), \quad x \in [0,D] \tag{5}$$

$$u(D,t) = U(t), \tag{6}$$

which transforms system (2) to a new system of the form[1]

$$\dot{Z}(t) = F\left(Z(t), U_t, U(t)\right), \tag{7}$$

where the function U_t is defined by $U_t(s) = U(t+s)$, for all $s \in [-D, 0]$. The control law that stabilizes system (7)[2] is given for all $t \geq 0$ by

$$U(t) = \kappa\left(Z(t), U_t\right). \tag{8}$$

Although the predictor-based design (8), (4), (3) is derived by employing the design methodology developed in Ponomarev (in press), in this article we introduce novel stability analysis tools, which, in comparison with Ponomarev (in press), allows one to remove the plant and controller growth restrictions, as well as the requirements that the control be affine and that the nominal controller achieves exponential stability of the transformed system. Specifically, we prove global asymptotic stability for systems that are not necessarily affine in the control, without necessarily imposing a linear growth condition either on the vector field or the nominal controller and without assuming that the nominal control law achieves exponential stability. Our stability analysis is based on estimates on closed-loop solutions.

We also identify a class of systems that includes systems transformable to a completely delay-free equivalent and which we categorize into two different types of systems. For this class of systems we also construct a novel Lyapunov functional with the aid of which we prove global asymptotic stability of the closed-loop system, thus providing an alternative stability proof.

[1] For the sake of clarity of presentation the exact form of F is given in Section 2.

[2] The specific properties of the closed-loop system and κ are specified in Section 2.

Although in order to help the reader to better understand the conceptual ideas of our methodology we concentrate on systems of the form (2), i.e., without distributed delay terms, the same tools can be applied to systems with distributed delay terms of the form

$$\dot{X}(t) = f\Bigg(X(t), U(t-D), \int_{t-D}^{t} b_1(\theta - t)U(\theta)d\theta, \ldots, \int_{t-D}^{t} b_m(\theta - t)U(\theta)d\theta, U(t)\Bigg). \tag{9}$$

1.3. Organization

In Section 2 we prove global asymptotic stability under predictor-based feedback for general nonlinear systems. In Section 3 we identify a class of systems that includes systems transformable to a delay-free equivalent. For this class of systems we construct a Lyapunov functional with the aid of which we prove global asymptotic stability under predictor-based feedback in Section 4. We illustrate the fact that the developed approach can be applied to systems with distributed delay terms in Section 5.

Notation: We use the common definition of class $\mathcal{K}$, $\mathcal{K}_\infty$, and $\mathcal{KL}$ functions from Khalil (2002). For an n-vector, the norm $|\cdot|$ denotes the usual Euclidean norm. For a function $u : [0,D] \times \mathbb{R}_+ \to \mathbb{R}$ we denote by $\|u(t)\|_\infty$ its spatial supremum norm, i.e., $\|u(t)\|_\infty = \sup_{x\in[0,D]} |u(x,t)|$. For any $c > 0$, we denote the spatially weighted supremum norm of u by $\|u(t)\|_{c,\infty} = \sup_{x\in[0,D]} |e^{cx}u(x,t)|$. For a vector valued function $p : [0,D] \times \mathbb{R}_+ \to \mathbb{R}^n$ we use a spatial supremum norm $\|p(t)\|_\infty = \sup_{x\in[0,D]} \sqrt{p_1(x,t)^2 + \cdots p_n(x,t)^2}$. For a function $U : [-D, \infty) \to \mathbb{R}, \forall t \geq 0$, the function U_t is defined by $U_t(s) = U(t+s), \forall s \in [-D, 0]$. We denote by $C^j(A; E)$ the space of functions that take values in E and have continuous derivatives of order j on A.

Solutions: We assume that the initial condition $U_0 \in C([-D,0];\mathbb{R})$ is compatible with the feedback law (8), i.e., it holds that $U_0(0) = \kappa\left(Z(0), U_0\right)$, such that under the assumptions that $\kappa : \mathbb{R}^n \times C([-D,0];\mathbb{R}) \to \mathbb{R}$ is locally Lipschitz and that $f : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^n$ is twice continuously differentiable (Assumption 1 in Section 2), which allows one to conclude that $F : \mathbb{R}^n \times C([-D,0];\mathbb{R}) \times \mathbb{R} \to \mathbb{R}^n$ is locally Lipschitz,[3] there exists a unique solution $Z(t) \in C^1([0,\infty), \mathbb{R}^n)$ and $U(t) \in C([0,\infty), \mathbb{R})$ (see Hale & Verduyn Lunel, 1993; Karafyllis, Pepe, & Jiang, 2009; Mazenc et al., 2012; Pepe, 2007; Pepe, Karafyllis, & Jiang, 2008),[4] which in turn implies from (2) that there exists a unique solution $X(t) \in C^1([0,\infty), \mathbb{R}^n)$.

2. Stability analysis for general systems

Assumption 1. The vector field $f : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^n$ is twice continuously differentiable with $f(0,0,0) = 0$ and satisfies

$$f(X, \omega, \Omega) - f(X, \omega, 0) = g(X, \Omega) \tag{10}$$

for all $(X, \omega, \Omega)^T \in \mathbb{R}^{n+2}$ and some $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$.

[3] The Lipschitzness of F (Lemma 1 in Section 2) follows by the regularity of f and the Lipschitzness of the solutions to $p_x(x) = f\left(p(x), u(x), 0\right), p(D) = Z$ with respect to $Z \in \mathbb{R}^n$ and $u \in C([0,D];\mathbb{R})$, as well as to $r_x(x) = \frac{\partial f(p(x),u(x),0)}{\partial p} r(x)$, $r(0) = g$ with respect to $g \in \mathbb{R}^n$, $p \in C([0,D];\mathbb{R}^n)$ and $u \in C([0,D];\mathbb{R})$ (see, e.g., Hale & Verduyn Lunel, 1993; Khalil, 2002).

[4] The fact that $Z(t)$ and $U(t)$ are defined on $[0,\infty)$ follows from the stability properties of system (7), (8), which are established employing Assumption 3 in Section 2.

An example of systems that satisfy Assumption 1 is nonlinear systems that are affine only in the non-delayed control variable, only in the delayed control variable, or in both.

For the sake of clarity, before presenting the additional assumptions on system (2) and the main result of this section, we first state the following lemma whose proof can be found in the Appendix.

Lemma 1. *Under Assumption 1, the transformation Z of the state X defined by (3)–(6) transforms system (2) to system (7), where*

$$\begin{aligned} F\left(Z(t), U_t, U(t)\right) = {} & f(Z(t), U(t), 0) \\ & + \Phi(D, 0, t) g\left(p(0, t), U(t)\right), \end{aligned} \tag{11}$$

and Φ denotes the state transition matrix associated with the following ODE in x (parametrized in t)

$$r_x(x, t) = \frac{\partial f\left(p(x, t), u(x, t), 0\right)}{\partial p} r(x, t). \tag{12}$$

One should notice that F is a function of $Z(t), U(t), U_t$ since p satisfies the following ODE in x parametrized in t

$$p_x(x, t) = f\left(p(x, t), u(x, t), 0\right) \tag{13}$$

$$p(D, t) = Z(t). \tag{14}$$

We point out that in the absence of Assumption 1 the vector field g in (11) would depend explicitly on the delayed input $U(t - D)$, thus canceling the effect of the Z transformation.

Assumption 2. The plant $\dot{X} = f(X, \omega, 0)$ is complete with respect to ω.

System $\dot{X} = f(X, \omega, 0)$ is forward complete if for every initial condition $X(0)$ and for every measurable locally bounded input signal ω the corresponding solution is defined for all $t \geq 0$, i.e., the maximal interval of existence is $[0, T^+_{\max})$ with $T^+_{\max} = +\infty$. It is backward complete if for every initial condition $X(0)$ and for every measurable locally bounded input signal ω the corresponding solution is defined for all $t \leq 0$, i.e., the maximal interval of existence is $(T^-_{\max}, 0]$ with $T^-_{\max} = -\infty$. It is complete when it is both forward and backward complete (see, for example, Lin, Sontag, & Wang, 1996).

The forward completeness requirement in Assumption 2 guarantees that the Z transformation $X \mapsto Z$, defined as $Z = p(D)$ via (3) (or, equivalently, via $p_x(x) = f(p(x), u(x), 0)$, $p(0) = X$) is globally well-defined. Analogously, the backward completeness requirement in Assumption 2 guarantees that the inverse X transformation $Z \mapsto X$, defined as $X = p(0)$ via (13), (14) is globally well-defined as well.

Assumption 3. There exist a locally Lipschitz feedback law $\kappa : \mathbb{R}^n \times C([-D, 0]; \mathbb{R}) \to \mathbb{R}$, a class $\mathcal{K}_\infty$ function $\hat{\rho}$, and a class $\mathcal{KL}$ function $\hat{\sigma}$ such that

$$|\kappa(Z, \phi)| \leq \hat{\rho}(|Z|) \tag{15}$$

for all $Z \in \mathbb{R}^n$ and $\phi \in C([-D, 0]; \mathbb{R})$, and for the closed-loop system $\dot{Z}(t) = F(Z(t), U_t, U(t))$, $U(t) = \kappa(Z(t), U_t)$ it holds that

$$|Z(t)| \leq \hat{\sigma}(|Z(0)|, t), \quad t \geq 0. \tag{16}$$

Theorem 1. *Consider the closed-loop system consisting of the plant (2) and the control law (8), (3), (4). Under Assumptions 1–3 there exists a class $\mathcal{KL}$ function $\hat{\beta}$ such that*

$$\Gamma(t) \leq \hat{\beta}(\Gamma(0), t), \quad t \geq 0, \tag{17}$$

where

$$\Gamma(t) = |X(t)| + \sup_{t-D \leq \theta \leq t} |U(\theta)|. \tag{18}$$

The proof of Theorem 1 is based on the following lemma whose proof can be found in the Appendix.

Lemma 2. *There exist class $\mathcal{K}_\infty$ functions ρ_1 and ρ_2 such that the following hold for all $t \geq 0$*

$$\|p(t)\|_\infty \leq \rho_1\left(|X(t)| + \|u(t)\|_\infty\right) \tag{19}$$

$$\|p(t)\|_\infty \leq \rho_2\left(|Z(t)| + \|u(t)\|_\infty\right). \tag{20}$$

Proof of Theorem 1. Using (15), (8) we get that

$$\sup_{t-D \leq \theta \leq t} |U(\theta)| \leq \hat{\rho}\left(\hat{\sigma}\left(|Z(0)|, t - D\right)\right), \quad t \geq D. \tag{21}$$

Moreover,

$$\begin{aligned} & \sup_{t-D \leq \theta \leq t} |U(\theta)| \leq \sup_{-D \leq \theta \leq 0} |U(\theta)| + \sup_{0 \leq \theta \leq t} |U(\theta)|, \\ & 0 \leq t \leq D. \end{aligned} \tag{22}$$

Combining (21), (22), by Lemma 2 (relation (19)), relation (4), and the fact that $u(x, t) = U(t + x - D)$ (which follows from (5), (6)) we get that

$$\begin{aligned} & \sup_{t-D \leq \theta \leq t} |U(\theta)| \leq \hat{\sigma}_1\left(|X(0)| + \sup_{-D \leq \theta \leq 0} |U(\theta)|, t\right), \\ & t \geq 0, \end{aligned} \tag{23}$$

where the class $\mathcal{KL}$ function $\hat{\sigma}_1$ is given by

$$\begin{aligned} \hat{\sigma}_1(s, t) = {} & \hat{\rho}\left(\hat{\sigma}\left(\rho_1(s), \max\{t - D, 0\}\right)\right) \\ & + s e^{-\lambda \max\{t-D, 0\}}, \end{aligned} \tag{24}$$

for an arbitrary $\lambda > 0$. Similarly, by Lemma 2 (relation (20)) and using the fact that $p(0, t) = X(t)$ we get that

$$|X(t)| \leq \hat{\sigma}_2\left(|X(0)| + \sup_{-D \leq \theta \leq 0} |U(\theta)|, t\right), \quad t \geq 0, \tag{25}$$

where the class $\mathcal{KL}$ function $\hat{\sigma}_2$ is defined as

$$\hat{\sigma}_2(s, t) = \rho_2\left(\hat{\sigma}\left(\rho_1(s), t\right) + \hat{\sigma}_1(s, t)\right). \tag{26}$$

Combining (25) and (26) we arrive at (17) with

$$\hat{\beta}(s, t) = \hat{\sigma}_1(s, t) + \hat{\sigma}_2(s, t). \quad \square \tag{27}$$

Example 1. We consider the following system, which is not affine in the control

$$\dot{X}_1(t) = 2X_2(t) + U(t) \tag{28}$$

$$\dot{X}_2(t) = \frac{X_2(t) + U(t - D)}{U(t - D)^2 + 1}, \tag{29}$$

and which satisfies Assumptions 1 and 2. Employing (7), (11) we get the transformed Z system as

$$\dot{Z}_1(t) = 2Z_2(t) + U(t) \tag{30}$$

$$\dot{Z}_2(t) = \frac{Z_2(t) + U(t)}{U(t)^2 + 1}. \tag{31}$$

System (30), (31) can be globally asymptotically stabilized with the control law

$$U(t) = -2Z_2(t) - Z_1(t), \tag{32}$$

which can be seen using the Lyapunov functional $\bar{V} = \frac{1}{2}\left(Z_1^2 + Z_2^2\right)$, which satisfies along the solutions of (30)–(32) $\dot{\bar{V}} \leq -\frac{1}{2}Z_1^2 - \frac{1}{2}\frac{Z_2^2}{(Z_1(t)+2Z_2(t))^2+1}$. Therefore, system (28), (29) can be stabilized with the control law (32), where Z_1, Z_2 are defined explicitly in terms of the plant and the actuator states as

$$Z_1(t) = X_1(t) + 2\int_{t-D}^{t} e^{\int_{t-D}^{\theta}\frac{ds}{U(s)^2+1}} d\theta X_2(t) + 2\int_{t-D}^{t}\left(\int_{t-D}^{\theta} e^{\int_{s}^{\theta}\frac{dr}{U(r)^2+1}}\frac{U(s)ds}{U(s)^2+1}\right)d\theta \quad (33)$$

$$Z_2(t) = e^{\int_{t-D}^{t}\frac{d\theta}{U(\theta)^2+1}} X_2(t) + \int_{t-D}^{t} e^{\int_{\theta}^{t}\frac{ds}{U(s)^2+1}}\frac{U(\theta)d\theta}{U(\theta)^2+1}. \quad (34)$$

Note that the transformed system (30), (31) is completely delay-free. In fact, system (28), (29) belongs to a larger class of systems that can be transformed to a delay-free equivalent. We recognize such a class of systems, which we categorize in two different types of systems, in Section 3 (system (28), (29) is of the type treated in Section 3.2).

3. Systems transformable to a delay-free equivalent

3.1. Systems of type I

We consider the following system

$$\dot{X}_1(t) = f_1\left(X_2(t), U(t)\right) + g_1\left(U(t-D)\right) \quad (35)$$

$$\dot{X}_2(t) = f_2\left(X_2(t), U(t)\right), \quad (36)$$

where $X_1 \in \mathbb{R}^{n_1}, X_2 \in \mathbb{R}^{n_2}$ is state, $U \in \mathbb{R}$ is control input, $D > 0$ is a delay, $t \in \mathbb{R}$ is time, $f_1 : \mathbb{R}^{n_2}\times\mathbb{R} \to \mathbb{R}^{n_1}$ and $f_2 : \mathbb{R}^{n_2}\times\mathbb{R} \to \mathbb{R}^{n_2}$ are twice continuously differentiable vector fields with $f_1(0,0) = 0$ and $f_2(0,0) = 0$, respectively, and $g_1 : \mathbb{R} \to \mathbb{R}^{n_1}$ is a continuously differentiable input vector field that satisfies $g_1(0) = 0$. For the sake of clarity, before presenting the main result of this section, we first state the following lemma whose proof can be found in the Appendix.

Lemma 3. *The transformation Z of the state X defined as*

$$p_1(x,t) = X_1(t) + \int_0^x \left(f_1\left(p_2(y,t),0\right) + g_1\left(u(y,t)\right)\right)dy, \quad x \in [0,D] \quad (37)$$

$$p_2(x,t) = X_2(t) + \int_0^x f_2\left(p_2(y,t),0\right)dy, \quad x \in [0,D] \quad (38)$$

$$Z_1(t) = p_1(D,t) \quad (39)$$

$$Z_2(t) = p_2(D,t), \quad (40)$$

where u is defined in (5), (6), transforms system (35), (36) to the following, delay-free system

$$\dot{Z}_1(t) = F_1\left(Z_2(t), U(t)\right) \quad (41)$$

$$\dot{Z}_2(t) = F_2\left(Z_2(t), U(t)\right), \quad (42)$$

where

$$F_1\left(Z_2(t), U(t)\right) = f_1(Z_2(t),0) + g_1\left(U(t)\right) + \int_0^D \frac{\partial f_1\left(p_2(y,t),0\right)}{\partial p_2}\Phi_2(y,0,t)dy \times G_2\left(p_2(0,t),U(t)\right) + G_1\left(p_2(0,t),U(t)\right) \quad (43)$$

$$F_2\left(Z_2(t), U(t)\right) = f_2\left(Z_2(t),0\right) + \Phi_2(D,0,t) \times G_2\left(p_2(0,t),U(t)\right), \quad (44)$$

with

$$G_1\left(p_2(0,t),U(t)\right) = f_1\left(p_2(0,t),U(t)\right) - f_1\left(p_2(0,t),0\right) \quad (45)$$

$$G_2\left(p_2(0,t),U(t)\right) = f_2\left(p_2(0,t),U(t)\right) - f_2\left(p_2(0,t),0\right), \quad (46)$$

and Φ_2 denotes the state transition matrix associated with the following ODE in x (parametrized in t)

$$r_{2x}(x,t) = \frac{\partial f_2\left(p_2(x,t),0\right)}{\partial p_2} r_2(x,t). \quad (47)$$

One should notice that F_1 and F_2 are functions only of Z_2 since G_1, G_2, and Φ_2 depend only on p_2, which satisfies the following ODE in x parametrized in t

$$p_{2x}(x,t) = f_2\left(p_2(x,t),0\right) \quad (48)$$

$$p_2(D,t) = Z_2(t). \quad (49)$$

The control law for system (35), (36) is given for all $t \geq 0$ by

$$U(t) = \kappa(Z(t)), \quad (50)$$

where $\kappa : \mathbb{R}^{n_1+n_2} \to \mathbb{R}$ is a continuously differentiable feedback law that satisfies $\kappa(0) = 0$.

Theorem 2. *Let the system $\dot{\Xi} = f_2(\Xi, 0)$ be complete and the system $\dot{\Xi} = F(\Xi, \kappa(\Xi))$ be globally asymptotically stable. There exists a class $\mathcal{KL}$ function β such that for the closed-loop system consisting of the plant (35), (36) and the control law (50), (37)–(40) the following holds*

$$\Gamma(t) \leq \beta\left(\Gamma(0), t\right), \quad t \geq 0, \quad (51)$$

where Γ is defined in (18).

The proof of Theorem 2 is based on the following lemma whose proof can be found in the Appendix.

Lemma 4. *There exist class $\mathcal{K}_\infty$ functions $\alpha_1, \alpha_2, \alpha_3$, and α_4 such that the following hold for all $t \geq 0$*

$$\|p_1(t)\|_\infty \leq \alpha_1\left(|X(t)| + \|u(t)\|_\infty\right) \quad (52)$$

$$\|p_2(t)\|_\infty \leq \alpha_2\left(|X_2(t)|\right) \quad (53)$$

$$\|p_1(t)\|_\infty \leq \alpha_3\left(|Z(t)| + \|u(t)\|_\infty\right) \quad (54)$$

$$\|p_2(t)\|_\infty \leq \alpha_4\left(|Z_2(t)|\right). \quad (55)$$

Proof of Theorem 2. Since the system $\dot{Z} = F(Z, \kappa(Z))$ is globally asymptotically stable there exists a class $\mathcal{KL}$ function σ such that

$$|Z(t)| \leq \sigma\left(|Z(0)|, t\right), \quad t \geq 0. \quad (56)$$

Hence, using the fact that κ is locally Lipschitz with $\kappa(0) = 0$, which implies that there exists a class $\mathcal{K}_\infty$ function $\hat{\alpha}$ such that $|\kappa(Z)| \leq \hat{\alpha}(|Z|)$, and (50) we get that

$$\sup_{t-D\leq\theta\leq t} |U(\theta)| \leq \hat{\alpha}\left(\sigma\left(|Z(0)|, t-D\right)\right), \quad t \geq D. \quad (57)$$

Moreover,

$$\sup_{t-D\leq\theta\leq t} |U(\theta)| \leq \sup_{-D\leq\theta\leq 0} |U(\theta)| + \sup_{0\leq\theta\leq t} |U(\theta)|, \quad 0 \leq t \leq D. \quad (58)$$

Combining (57), (58), by Lemma 4 (relations (52), (53)), relations (39), (40), and the fact that $u(x,t) = U(t+x-D)$ (which follows from (5), (6)) we get that

$$\sup_{t-D\leq\theta\leq t} |U(\theta)| \leq \sigma_1\left(|X(0)| + \sup_{-D\leq\theta\leq 0} |U(\theta)|, t\right), \quad t \geq 0, \quad (59)$$

where the class $\mathcal{KL}$ function σ_1 is given by

$$\sigma_1(s,t) = \hat{\alpha}\left(\sigma\left(\alpha_1(s) + \alpha_2(s), \max\{t-D, 0\}\right)\right) + se^{-\lambda\max\{t-D,0\}}, \quad (60)$$

for an arbitrary $\lambda > 0$. Analogously, by Lemma 4 (relations (54), (55)) and the fact that $p_1(0,t) = X_1(t), p_2(0,t) = X_2(t)$ we get that

$$|X(t)| \le \sigma_2\left(|X(0)| + \sup_{-D\le\theta\le 0}|U(\theta)|, t\right), \quad t \ge 0, \tag{61}$$

where the class $\mathcal{KL}$ function σ_2 is defined as

$$\begin{aligned}\sigma_2(s,t) &= \alpha_3\left(\sigma\left(\alpha_1(s)+\alpha_2(s), t\right) + \sigma_1(s,t)\right)\\ &\quad + \alpha_4\left(\sigma\left(\alpha_1(s)+\alpha_2(s), t\right) + \sigma_1(s,t)\right).\end{aligned} \tag{62}$$

Combining (61) and (62) we arrive at (51) with

$$\beta(s,t) = \sigma_1(s,t) + \sigma_2(s,t). \quad \square \tag{63}$$

Example 2. We consider the following system

$$\dot{X}_1(t) = X_2(t) - X_2(t)^2 U(t) - U(t-1) \tag{64}$$

$$\dot{X}_2(t) = U(t). \tag{65}$$

Employing transformation (37)–(40), which is written for system (64), (65) as

$$Z_1(t) = X_1(t) + X_2(t) - \int_{t-1}^{t} U(\theta)d\theta \tag{66}$$

$$Z_2(t) = X_2(t), \tag{67}$$

we obtain

$$\dot{Z}_1(t) = Z_2(t) - Z_2(t)^2 U(t) \tag{68}$$

$$\dot{Z}_2(t) = U(t). \tag{69}$$

System (68), (69) can be stabilized with the following control law (Krstic, 2004)

$$U(t) = -Z_1(t) - 2Z_2(t) - \frac{1}{3}Z_2(t)^3. \tag{70}$$

Thus, the control law for system (64), (65) is given by

$$U(t) = -X_1(t) - 3X_2(t) - \frac{1}{3}X_2(t)^3 + \int_{t-1}^{t} U(\theta)d\theta. \tag{71}$$

In Fig. 1 we show the response of system (64), (65) under the control law (71) and the corresponding control effort for initial conditions $X_1(0) = X_2(0) = 1$ and $U(\theta) = 0$, for all $-1 \le \theta \le 0$. As Theorem 2 predicts, the closed-loop system is asymptotically stable.

Note that one could arrive at the same control law by first linearizing system (64), (65) with the change of variables (Krstic, 2004) $\zeta_1 = X_1 + X_2 + \frac{1}{3}X_2^3$, $\zeta_2 = X_2$, resulting in the following system

$$\dot{\zeta}_1(t) = \zeta_2(t) + U(t) - U(t-1) \tag{72}$$

$$\dot{\zeta}_2(t) = U(t), \tag{73}$$

which can be stabilized employing the control law (Artstein, 1982)

$$U(t) = -Z_1(t) - Z_2(t) \tag{74}$$

$$Z_1(t) = \zeta_1(t) + \zeta_2(t) - \int_{t-1}^{t} U(\theta)d\theta \tag{75}$$

$$Z_2(t) = \zeta_2(t). \tag{76}$$

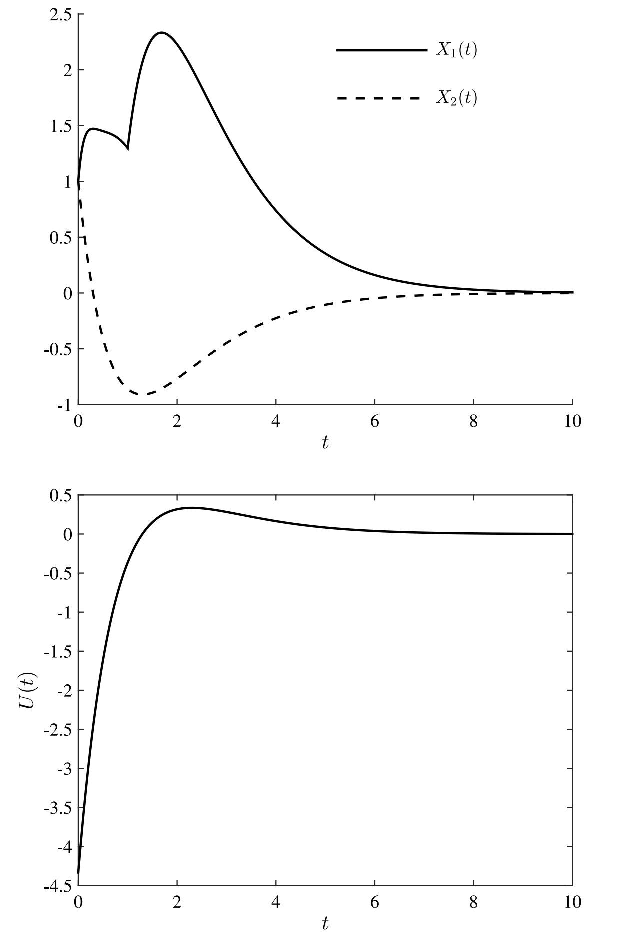

Fig. 1. The response of system (64), (65) (top) under the control law (71) and the corresponding control effort (bottom) for initial conditions $X_1(0) = X_2(0) = 1$ and $U(\theta) = 0$, for all $-1 \le \theta \le 0$.

3.2. Systems of type II

Motivated by Example 1 we consider the following special class of systems of the form (2)

$$\dot{X}_1(t) = f_1\left(X_2(t), U(t-D)\right) + g_1\left(U(t)\right) \tag{77}$$

$$\dot{X}_2(t) = f_2\left(X_2(t), U(t-D)\right). \tag{78}$$

Note that in comparison with (35), (36) f_1 and f_2 now depend on the delayed rather than the current input, whereas g_1 depends on the current rather than the delayed input. It can be shown that the transformation

$$p_1(x,t) = X_1(t) + \int_0^x \left(f_1\left(p_2(y,t), u(y,t)\right)\right) dy, \quad x \in [0, D] \tag{79}$$

$$p_2(x,t) = X_2(t) + \int_0^x f_2\left(p_2(y,t), u(y,t)\right) dy, \quad x \in [0, D] \tag{80}$$

$$Z_1(t) = p_1(D,t) \tag{81}$$

$$Z_2(t) = p_2(D,t), \tag{82}$$

transforms system (77), (78) to the following, delay-free system

$$\dot{Z}_1(t) = \bar{F}_1\left(Z_2(t), U(t)\right) \tag{83}$$

$$\dot{Z}_2(t) = \bar{F}_2\left(Z_2(t), U(t)\right), \tag{84}$$

where

$$\bar{F}_1(Z_2(t), U(t)) = f_1(Z_2(t), U(t)) + g_1(U(t)) \tag{85}$$

$$\bar{F}_2(Z_2(t), U(t)) = f_2(Z_2(t), U(t)). \tag{86}$$

One can obtain the following result whose proof follows the same lines with the proof of Theorem 2.

Theorem 3. *Let the system $\dot{\Xi} = f_2(\Xi, \omega)$ be complete with respect to ω and the system $\dot{\Xi} = \bar{F}(\Xi, \kappa(\Xi))$ be globally asymptotically stable. There exists a class $\mathcal{KL}$ function β^* such that for the closed-loop system consisting of the plant (77), (78) and the control law (50), (79)–(82) the following holds*

$$\Gamma(t) \leq \beta^*(\Gamma(0), t), \quad t \geq 0, \tag{87}$$

where Γ is defined in (18).

4. Lyapunov-based stability analysis for systems of type I and II

4.1. Type I systems

Theorem 4. *Let the system $\dot{\Xi} = f_2(\Xi, 0)$ be complete and the system $\dot{\Xi} = F(\Xi, \kappa(\Xi))$ be globally asymptotically stable and backward complete. There exists a class $\mathcal{KL}$ function $\bar{\beta}$ such that for the closed-loop system consisting of the plant (35), (36) and the control law (50), (37)–(40) the following holds*

$$\Gamma(t) \leq \bar{\beta}(\Gamma(0), t), \quad t \geq 0, \tag{88}$$

where Γ is defined in (18).

Proof of Theorem 4. Consider the new PDE state w defined as

$$w(x,t) = u(x,t) - \kappa(z(x,t)), \quad x \in [0, D], \tag{89}$$

where

$$z_x(x,t) = F_{cl}(z(x,t)), \quad x \in [0, D] \tag{90}$$

$$z(D,t) = Z(t) \tag{91}$$

$$F_{cl}(Z) = F(Z, \kappa(Z)), \tag{92}$$

with an initial condition satisfying $z_0'(x) = F_{cl}(z_0(x))$, $z_0(D) = Z(0)$. Using (90), (91) and the fact that $\dot{Z}(t) = F_{cl}(Z(t))$, for all $t \geq 0$, we get that

$$z_t(x,t) - z_x(x,t) = -\int_x^D \frac{\partial F_{cl}(z(y,t))}{\partial z} \times \left(z_t(y,t) - z_y(y,t)\right) dy, \tag{93}$$

and hence, $z_t(x,t) = z_x(x,t)$, for all $t \geq 0$ and $x \in [0, D]$. Therefore, using the facts that $u_t = u_x$ and $u(D) = \kappa(Z)$ we obtain

$$w_t(x,t) = w_x(x,t), \quad x \in [0, D] \tag{94}$$

$$w(D,t) = 0. \tag{95}$$

Since the system $\dot{Z} = F_{cl}(Z)$ is globally asymptotically stable (Khalil, 2002) there exists a smooth function $S : \mathbb{R}^{n_1+n_2} \to \mathbb{R}_+$, class $\mathcal{K}_\infty$ functions $\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3$, and a class $\mathcal{K}$ function γ such that,

$$\bar{\alpha}_1(|Z|) \leq S(Z) \leq \bar{\alpha}_2(|Z|) \tag{96}$$

$$\frac{\partial S(Z)}{\partial Z} F_{cl}(Z) \leq -\gamma(|Z|), \tag{97}$$

for all $Z \in \mathbb{R}^{n_1+n_2}$. Using the fact that $\frac{d\|w(t)\|_{c,\infty}}{dt} \leq -c\|w(t)\|_{c,\infty}$ for any $c > 0$ (Theorem 5 from Krstic, 2010) and (96), (97) we get that the functional

$$V(t) = S(Z(t)) + \|w(t)\|_{c,\infty}, \tag{98}$$

satisfies along the solutions of system $\dot{Z}(t) = F_{cl}(Z(t))$, (94), (95)

$$\dot{V}(t) \leq -\gamma_1(V(t)), \quad t \geq 0, \tag{99}$$

for some class $\mathcal{K}$ function γ_1. With the comparison principle (see, e.g., Lemma 3.4 in Khalil, 2002) and Lemma 4.4 in Khalil (2002) we get that $V(t) \leq \bar{\beta}_1(V(0), t)$ for some class $\mathcal{KL}$ function $\bar{\beta}_1$. Thus, with the help of (96) we arrive at

$$|Z(t)| + \|w(t)\|_\infty \leq \bar{\beta}_2(|Z(0)| + \|w(0)\|_\infty, t). \tag{100}$$

From the backward completeness assumption of $\dot{Z} = F_{cl}(Z)$ it follows that system $z_\xi^{b}(\xi,t) = -F_{cl}\left(z^{b}(\xi,t)\right)$, where $z^{b}(\xi,t) = z(D-\xi,t)$ and $x = D-\xi$, is forward complete. Thus, using Lemma 3.5 in Karafyllis (2004) and the fact that $\|z^{b}(t)\|_\infty = \|z(t)\|_\infty$ one can conclude that there exists a class $\mathcal{K}_\infty$ function $\bar{\psi}$ such that

$$\|z(t)\|_\infty \leq \bar{\psi}(|Z(t)|). \tag{101}$$

Hence, by (89) it follows that

$$\|w(t)\|_\infty \leq \bar{\psi}_1(|Z(t)| + \|u(t)\|_\infty), \tag{102}$$

where the class $\mathcal{K}_\infty$ function $\bar{\psi}_1$ is given by $\bar{\psi}_1(s) = s + \hat{\alpha}\left(\bar{\psi}(s)\right)$. By Lemma 4 (relations (52), (53)) and relations (39), (40) we get that

$$|Z(t)| + \|w(t)\|_\infty \leq \bar{\psi}_2(|X(t)| + \|u(t)\|_\infty), \tag{103}$$

where $\bar{\psi}_2(s) = s + 2\alpha_1(s) + 2\alpha_2(s) + \hat{\alpha}\left(\bar{\psi}(s + \alpha_1(s) + \alpha_2(s))\right)$. Using (89) we arrive at

$$\|u(t)\|_\infty \leq \bar{\psi}_1(|Z(t)| + \|w(t)\|_\infty). \tag{104}$$

By Lemma 4 (relations (54), (55)) and the fact that $p_1(0,t) = X_1(t)$, $p_2(0,t) = X_2(t)$ we get from (104) that

$$|X(t)| + \|u(t)\|_\infty \leq \bar{\psi}_3(|Z(t)| + \|w(t)\|_\infty), \tag{105}$$

where, $\bar{\psi}_3(s) = \bar{\psi}_1(s) + \alpha_3\left(s + \bar{\psi}_1\right) + \alpha_4\left(s + \bar{\psi}_1\right)$. Combining (100), (103), (105) the proof is completed. □

4.2. Type II systems

Employing the same arguments with the proof of Theorem 4 we obtain the following result.

Theorem 5. *Let the system $\dot{\Xi} = f_2(\Xi, \omega)$ be complete with respect to ω and the system $\dot{\Xi} = \bar{F}(\Xi, \kappa(\Xi))$ be globally asymptotically stable and backward complete. There exists a class $\mathcal{KL}$ function $\bar{\sigma}$ such that for the closed-loop system consisting of the plant (77), (78) and the control law (50), (79)–(82) the following holds*

$$\Gamma(t) \leq \bar{\sigma}(\Gamma(0), t), \quad t \geq 0, \tag{106}$$

where Γ is defined in (18).

5. Conclusions and discussion

Although in order to help the reader to better digest the details of our methodology we presented in detail the case of nonlinear systems without distributed delay terms, one could consider nonlinear systems having the form (9).[5] We illustrate this fact by considering the following system

$$\dot{X}(t) = f\left(X(t), U(t-D), \int_{t-D}^{t} b(\theta - t)U(\theta)d\theta, U(t)\right), \quad (107)$$

where $b : [-D, 0] \to \mathbb{R}$ is continuously differentiable. The control law that globally asymptotically stabilizes (107) is given by (4), (8) where for all $x \in [0, D]$

$$p(x,t) = X(t) + \int_0^x f\left(p(y,t), u(y,t), \int_y^D b(r-y-D)u(r,t)dr, 0\right) dy, \quad (108)$$

under similar assumptions to Assumptions 1–3. Specifically, it is assumed that (i) the vector field f satisfies

$$f(X, \omega, \chi, \Omega) - f(X, \omega, \chi, 0) = g(X, \chi, \Omega), \quad (109)$$

for all $(X, \omega, \chi, \Omega)^T \in \mathbb{R}^{n+3}$ and some $g : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^n$, (ii) the system $\dot{X} = f(X, \omega, \chi, 0)$ is complete with respect to $(\omega, \chi)^T$, and (iii) Assumption 3 holds for some κ, where F is defined by

$$\begin{aligned} F =& f(Z(t), U(t), 0, 0) + \Phi(D, 0, t) \\ &\times g\left(p(0,t), \int_0^D b(y-D)u(y,t)d\theta, U(t)\right) \\ &+ \int_0^D \Phi(D, x, t) R\left(p(x,t), u(x,t), \int_x^D b(y-x-D)u(y,t)dy, 0\right) b(-x)dxU(t) \end{aligned} \quad (110)$$

$$R = \frac{\partial f\left(p(x), u(x), \int_x^D b(y-x-D)u(y)dy, 0\right)}{\partial \chi}, \quad (111)$$

and Φ denotes the state transition matrix associated with the following ODE in x (parametrized in t)

$$r_x(x) = \frac{\partial f\left(p(x), u(x), \int_x^D b(y-x-D)u(y)dy, 0\right)}{\partial p} r(x). \quad (112)$$

Moreover, one could extend the class of Type I systems by considering the following system

$$\dot{X}_1(t) = f_1(X_2(t), U(t)) + g_1(U(t-D)) + \int_{t-D}^{t} B(\theta - t)U(\theta)d\theta \quad (113)$$

$$\dot{X}_2(t) = f_2(X_2(t), U(t)), \quad (114)$$

under the control law (50), (4), (38), and

$$p_1(x,t) = X_1(t) + \int_0^x \left(f_1(p_2(y,t), 0) + g_1(u(y,t)) + \int_y^D B(r-y-D)u(r,t)dr\right) dy, \quad x \in [0, D]. \quad (115)$$

Global asymptotic stability of the closed-loop system (113), (114), (50), (4), (38), and (115) can be proved under the assumptions that (i) $\dot{\Xi}_2 = f_2(\Xi_2, 0)$ is complete and (ii) that $\dot{\Xi} = F(\Xi, \kappa(\Xi))$ is globally asymptotically stable, where F_2 is defined in (44) and

$$\begin{aligned} F_1(Z_2(t), U(t)) =& f_1(Z_2(t), 0) + g_1(U(t)) \\ &+ \int_0^D \frac{\partial f_1(p_2(x,t), 0)}{\partial p_2} \Phi_2(x, 0, t)dx \\ &\times G_2(p_2(0,t), U(t)) + G_1(p_2(0,t), \\ &U(t)) + \int_0^D B(-x)dxU(t), \end{aligned} \quad (116)$$

where G_1, G_2, and Φ_2 are defined in Lemma 3.

Note that an analogous extension for Type II systems would not be possible because in this case, the vector field $\bar{F}_1$ defined in (85) would also depend on U_t rather than only on $Z_2(t)$ and $U(t)$.

Appendix

Proof of Lemma 1

Differentiating (3) with respect to t and x and using (2), (5), (6) (which also imply that $u(0,t) = U(t-D)$) we get that

$$\begin{aligned} p_t(x,t) - p_x(x,t) =& \int_0^x \frac{\partial f(p(y,t), u(y,t), 0)}{\partial p} \\ &\times \left(p_t(y,t) - p_y(y,t)\right) dy \\ &+ f(p(0,t), u(0,t), u(D,t)) \\ &- f(p(0,t), u(0,t), 0). \end{aligned} \quad (A.1)$$

It follows from (A.1) that

$$\begin{aligned} p_t(x,t) =& p_x(x,t) + \Phi(x, 0, t)\left(f(p(0,t), u(0,t), u(D,t))\right. \\ &\left. - f_2(p(0,t), u(0,t), 0)\right). \end{aligned} \quad (A.2)$$

Hence, using definition (4) and differentiating (3) with respect to x we get (7)–(11).

Proof of Lemma 2

We first prove (19). From the forward completeness assumption of system $\dot{\Xi} = f(\Xi, \omega, 0)$ and the fact that p satisfies the following ODE in x

$$p_x(x,t) = f(p(x,t), u(x,t), 0), \quad x \in [0, D] \quad (A.3)$$

$$p(0,t) = X_2(t), \quad (A.4)$$

we get estimate (19), employing, for example, Lemma 3.5 in Karafyllis (2004). We prove next (20). Performing the change of variables $y = D - x$ in (A.3), (A.4) and using definition (4) we conclude that the signal $p^{b}(y,t) = p(D-y,t)$ satisfies the following initial value problem

$$p^{b}_y(y,t) = -f\left(p^{b}(y,t), u^{b}(y,t), 0\right), \quad y \in [0, D] \quad (A.5)$$

$$p^{b}(0,t) = Z(t). \quad (A.6)$$

From the backward completeness assumption it follows that (A.5) is forward complete, and hence, using the fact that $\|p(t)\|_\infty = \|p^{b}(t)\|_\infty$ we get (20).

Proof of Lemma 3

Differentiating (37), (38) with respect to t and x and using (5), (6) (which also imply that $u(0,t) = U(t-D)$), (35), (36) we get that

$$\begin{aligned} p_{1t}(x,t) - p_{1x}(x,t) =& \int_0^x \frac{\partial f_1(p_2(y,t), 0)}{\partial p_2} \\ &\times \left(p_{2t}(y,t) - p_{2y}(y,t)\right) dy \end{aligned}$$

[5] Note that the form (9) implies that for affine systems of the form (1) the input vector field B_{int} should be of the form $B_{\text{int}}(\theta, X) = h_1(X)b_1(\theta) + h_2(X)b_2(\theta) + \cdots + h_m(X)b_m(\theta)$, for some continuously differentiable $h_i : \mathbb{R}^n \to \mathbb{R}^n$ and $b_i : [-D, 0] \to \mathbb{R}$, $i = 1, \ldots, m$.

$$+f_1\left(p_2(0,t),u(D,t)\right) - f_1\left(p_2(0,t),0\right) \tag{A.7}$$

$$p_{2t}(x,t) - p_{2x}(x,t) = \int_0^x \frac{\partial f_2\left(p_2(y,t),0\right)}{\partial p_2} \times \left(p_{2t}(y,t) - p_{2y}(y,t)\right)dy + f_2\left(p_2(0,t),u(D,t)\right) - f_2\left(p_2(0,t),0\right). \tag{A.8}$$

It follows from (A.7), (A.8) that

$$p_{1t}(x,t) = p_{1x}(x,t) + \int_0^x \frac{\partial f_1\left(p_2(y,t),0\right)}{\partial p_2}\Phi_2(y,0,t)dy \times \left(f_2\left(p_2(0,t),u(D,t)\right) - f_2\left(p_2(0,t),0\right)\right) + f_1\left(p_2(0,t),u(D,t)\right) - f_1\left(p_2(0,t),0\right) \tag{A.9}$$

$$p_{2t}(x,t) = p_{2x}(x,t) + \Phi_2(x,0,t)\left(f_2\left(p_2(0,t),u(D,t)\right) - f_2\left(p_2(0,t),0\right)\right). \tag{A.10}$$

Hence, using definition (39), (40) and differentiating (37), (38) with respect to x we get (41)–(46).

Proof of Lemma 4

We first prove (53), (52). From the forward completeness assumption of system $\dot{\Xi} = f_2(\Xi, 0)$ and the fact that p_2 satisfies the following ODE in x

$$p_{2x}(x,t) = f_2\left(p_2(x,t),0\right), \quad x \in [0,D] \tag{A.11}$$

$$p_2(0,t) = X_2(t), \tag{A.12}$$

we get estimate (53) by using, for example, Lemma 3.5 in Karafyllis (2004). Using (37) we get that

$$|p_1(x,t)| \le |X_1(t)| + D\hat{\alpha}_1\left(\|p_2(t)\|_\infty\right) + D\hat{\alpha}_2\left(\|u(t)\|_\infty\right), \tag{A.13}$$

where we used the fact that f_1 and g_1 are locally Lipschitz with $f_1(0,0) = 0$ and $g_1(0) = 0$, respectively, which allows one to conclude that there exist class $\mathcal{K}_\infty$ functions $\hat{\alpha}_1$, $\hat{\alpha}_2$ such that $|f_1(X_2,0)| \le \hat{\alpha}_1(|X_2|)$ and $|g_1(U)| \le \hat{\alpha}_2(|U|)$, for all $(X_2,U)^T \in \mathbb{R}^{n_2+1}$. Estimate (52) follows using (53). We prove next (55), (54). Performing the change of variables $y = D - x$ in (A.11), (A.12) and using definition (40) we conclude that the signal $p_2^b(y,t) = p_2(D-y,t)$ satisfies the following initial value problem

$$p_{2y}^b(y,t) = -f_2\left(p_2^b(y,t),0\right), \quad y \in [0,D] \tag{A.14}$$

$$p_2^b(0,t) = Z_2(t). \tag{A.15}$$

From the backward completeness assumption it follows that (A.14) is forward complete, and hence, using the fact that $\|p_2(t)\|_\infty = \|p_2^b(t)\|_\infty$ we get (55). Since from (37), (39) it follows that

$$p_1(x,t) = Z_1(t) - \int_x^{D_1}\left(f_1\left(p_2(y,t),0\right) + g_1\left(u(y,t)\right)\right) \times dy, \quad x \in [0,D], \tag{A.16}$$

estimate (54) follows.

References

Ariola, M., & Pironti, A. (2008). H_∞ optimal terminal control for linear systems with delayed states and controls. *Automatica*, *44*, 2676–2679.
Artstein, Z. (1982). Linear systems with delayed controls: A reduction. *IEEE Transactions on Automatic Control*, *27*, 869–879.
Bekiaris-Liberis, N., & Krstic, M. (2011). Lyapunov stability of linear predictor feedback for distributed input delay. *IEEE Transactions on Automatic Control*, *56*, 655–660.
Bekiaris-Liberis, N., & Krstic, M. (2013a). Compensation of state-dependent input delay for nonlinear systems. *IEEE Transactions on Automatic Control*, *58*, 275–289.
Bekiaris-Liberis, N., & Krstic, M. (2013b). *Nonlinear control under nonconstant delays*. SIAM.
Bekiaris-Liberis, N., & Krstic, M. (2013c). Robustness of nonlinear predictor feedback laws to time- and state-dependent delay perturbations. *Automatica*, *49*, 1576–1590.
Bresch-Pietri, D., Chauvin, J., & Petit, N. (2014). Prediction-based stabilization of linear systems subject to input-dependent input delay of integral type. *IEEE Transactions of Automatic Control*, *59*, 2385–2399.
Bresch-Pietri, D., & Krstic, M. (2014). Delay-adaptive control for nonlinear systems. *IEEE Transactions on Automatic Control*, *59*, 1203–1218.
Chen, W.-H., & Zheng, W. X. (2002). On improved robust stabilization of uncertain systems with unknown input delay. *Automatica*, *42*, 1067–1072.
Fridman, E. (2014). *Introduction to time-delay systems*. Birkhauser.
Goebel, G., Munz, U., & Allgower, F. (2010). Stabilization of linear systems with distributed input delay. In *Proceedings of the American control conference*.
Hale, J. K., & Verduyn Lunel, S. M. (1993). *Introduction to functional differential equations*. Springer.
Karafyllis, I. (2004). The non-uniform in time small-gain theorem for a wide class of control systems with outputs. *European Journal of Control*, *10*, 307–323.
Karafyllis, I. (2011). Stabilization by means of approximate predictors for systems with delayed input. *SIAM Journal on Control and Optimization*, *49*, 1100–1123.
Karafyllis, I., & Krstic, M. (2012). Nonlinear stabilization under sampled and delayed measurements, and with inputs subject to delay and zero-order hold. *IEEE Transactions on Automatic Control*, *57*, 1141–1154.
Karafyllis, I., & Krstic, M. (2013). Stabilization of nonlinear delay systems using approximate predictors and high-gain observers. *Automatica*, *49*, 3623–3631.
Karafyllis, I., & Krstic, M. (2014). Numerical schemes for nonlinear predictor feedback. *Mathematics of Control, Signals, and Systems*, *26*, 519–546.
Karafyllis, I., Pepe, P., & Jiang, Z.-P. (2009). Stability results for systems described by coupled retarded functional differential equations and functional difference equations. *Nonlinear Analysis*, *71*, 3339–3362.
Khalil, H. (2002). *Nonlinear systems* (3rd ed.). Prentice Hall.
Krstic, M. (2004). Feedback linearizability and explicit integrator forwarding controllers for classes of feedforward systems. *IEEE Transactions on Automatic Control*, *49*, 1668–1682.
Krstic, M. (2009). *Delay compensation for nonlinear, adaptive, and PDE systems*. Birkhauser.
Krstic, M. (2010). Input delay compensation for forward complete and feedforward nonlinear systems. *IEEE Transactions on Automatic Control*, *55*, 287–303.
Li, Z.-Y., Zhou, B., & Lam, J. (2014). Lyapunov-Krasovskii functionals for predictor feedback control of linear systems with multiple input delays. *Applied Mathematics and Computation*, *244*, 303–311.
Lin, Y., Sontag, E., & Wang, Y. (1996). A smooth converse Lyapunov theorem for robust stability. *SIAM Journal on Control and Optimization*, *34*, 124–160.
Manitius, A. Z., & Olbrot, A. W. (1979). Finite spectrum assignment problem for systems with delays. *IEEE Transactions on Automatic Control*, *24*, 541–552.
Marquez-Martinez, L. A., & Moog, C. H. (2004). Input–output feedback linearization of time-delay systems. *IEEE Transactions on Automatic Control*, *49*, 781–786.
Mazenc, F., & Malisoff, M. (2014). Local stabilization of nonlinear systems through the reduction model approach. *IEEE Transactions on Automatic Control*, *59*, 3033–3039.
Mazenc, F., & Niculescu, S.-I. (2011). Generating positive and stable solutions through delayed state feedback. *Automatica*, *47*, 525–533.
Mazenc, F., Niculescu, S.-I., & Bekaik, M. (2013). Stabilization of time-varying nonlinear systems with distributed delay by feedback of plant's state. *IEEE Transactions on Automatic Control*, *58*, 264–269.
Mazenc, F., Niculescu, S.-I. , &, & Krstic, M. (2012). Lyapunov-Krasovskii functionals and application to input delay compensation for linear time-invariant systems. *Automatica*, *48*, 1317–1323.
Mondie, S., & Michiels, W. (2003). Finite spectrum assignment of unstable time-delay systems with a safe implementation. *IEEE Transactions on Automatic Control*, *48*, 2207–2212.
Niculescu, S.-I. (2001). *Delay effects on stability: A robust control approach*. Springer.
Pepe, P. (2007). On LiapunovKrasovskii functionals under Caratheodory conditions. *Automatica*, *43*, 701–706.
Pepe, P., Karafyllis, I., & Jiang, Z.-P. (2008). On the LiapunovKrasovskii methodology for the ISS of systems described by coupled delay differential and difference equations. *Automatica*, *44*, 2266–2273.
Ponomarev, A. (2015). Nonlinear predictor feedback for input-affine systems with distributed input delays. *IEEE Transactions on Automatic Control*, http://dx.doi.org/10.1109/TAC.2015.2496191. in press.
Ponomarev, A. (2016). Reduction-based robustness analysis of linear predictor feedback for distributed input delay. *IEEE Transactions on Automatic Control*, *61*, 468–472. http://dx.doi.org/10.1109/TAC.2015.2437520.
Richard, J.-P. (2003). Time-delay systems:an overview of some recent advances and open problems. *Automatica*, *39*, 1667–1694.
Roesch, O., Roth, H., & Niculescu, S.-I. (2005). Remote control of mechatronic systems over communication networks. In *Proceedings of the int. conf. mechatron. autom*.
Shuai, L., Lihua, X., & Huanshui, Z. (2008). Linear quadratic regulation for linear continuous-time systems with distributed input delay. In *Proceedings of the Chinese control conference*.
Xia, X., Marquez-Martinez, L. A., Zagalak, P., & Moog, C. H. (2002). Analysis of nonlinear time-delay systems using modules over non-commutative rings. *Automatica*, *38*, 1549–1555.
Xie, L., Fridman, E., & Shaked, U. (2001). Robust control of distributed delay systems with application to combustion control. *IEEE Transactions on Automatic Control*, *46*, 1930–1935.

Yue, D. (2004). Robust stabilization of uncertain systems with unknown input delay. *Automatica*, *40*, 331–336.
Zheng, F., & Frank, P. M. (2002). Robust control of uncertain distributed delay systems with application to the stabilization of combustion in rocket motor chambers. *Automatica*, *38*, 487–497.
Zhou, B. (2014). Input delay compensation of linear systems with both state and input delays by nested prediction. *Automatica*, *50*, 1434–1443.

Nikolaos Bekiaris-Liberis received the Ph.D. degree from the University of California, San Diego, in 2013. From 2013 to 2014 he was a postdoctoral researcher at the University of California, Berkeley. Dr. Bekiaris-Liberis is currently a postdoctoral researcher at the Dynamic Systems & Simulation Laboratory, Technical University of Crete, Greece. He has coauthored the SIAM book *Nonlinear Control under Nonconstant Delays*. His interests are in delay systems, distributed parameter systems, nonlinear control, and their applications. Dr. Bekiaris-Liberis was a finalist for the student best paper award at the 2010 ASME Dynamic Systems and Control Conference and at the 2013 IEEE Conference on Decision and Control. He received the Chancellor's Dissertation Medal in Engineering from the University of California, San Diego, in 2014. Dr. Bekiaris-Liberis received the best paper award in the 2015 International Conference on Mobile Ubiquitous Computing, Systems, Services and Technologies.

Miroslav Krstic holds the Alspach endowed chair and is the founding director of the Cymer Center for Control Systems and Dynamics at UC San Diego. He also serves as Associate Vice Chancellor for Research at UCSD. As a graduate student, Krstic won the UC Santa Barbara best dissertation award and student best paper awards at CDC and ACC. Krstic is a Fellow of IEEE, IFAC, ASME, and IET (UK), and a Distinguished Visiting Fellow of the Royal Academy of Engineering. He has received the PECASE, NSF Career, and ONR Young Investigator awards, the Axelby and Schuck paper prizes, the Chestnut textbook prize, and the first UCSD Research Award given to an engineer. Krstic has held the Springer Visiting Professorship at UC Berkeley. He serves as Senior Editor in IEEE Transactions on Automatic Control and Automatica, as editor of two Springer book series, and has served as Vice President for Technical Activities of the IEEE Control Systems Society and as chair of the IEEE CSS Fellow Committee. Krstic has coauthored ten books on adaptive, nonlinear, and stochastic control, extremum seeking, control of PDE systems including turbulent flows, and control of delay systems.