Predictor-Feedback Stabilization of Multi-Input Nonlinear Systems

Nikolaos Bekiaris-Liberis and Miroslav Krstic

***Abstract*—We develop a predictor-feedback control design for multi-input nonlinear systems with distinct input delays, of arbitrary length, in each individual input channel. Due to the fact that different input signals reach the plant at different time instants, the key design challenge, which we resolve, is the construction of the predictors of the plant's state over distinct prediction horizons such that the corresponding input delays are compensated. Global asymptotic stability of the closed-loop system is established by utilizing arguments based on Lyapunov functionals or estimates on solutions. We specialize our methodology to linear systems for which the predictor-feedback control laws are available explicitly and for which global exponential stability is achievable. A detailed example is provided dealing with the stabilization of the nonholonomic unicycle, subject to two different input delays affecting the speed and turning rate, for the illustration of our methodology.**

***Index Terms*—Delay systems, distributed parameter systems, nonlinear systems, predictor feedback.**

I. Introduction

A. Background and Motivation

Despite the recent outburst in the development of predictor-based control laws for nonlinear systems with input delays [5]–[11], [13]–[17], [26]–[32], [35]–[37], [43]–[47], the problem of the systematic predictor-feedback stabilization of multi-input nonlinear systems with, potentially different, in each individual input channel, long input delays, has remained, heretofore, untackled, although the problem was solved in the linear case in the late 1970s [42] (see also [4], [38], and [57]). In this paper, we address the problem of stabilization of multi-input nonlinear systems with distinct input delays of arbitrary length and develop a nonlinear version of the prediction-based control laws developed in [42] and recently in [55] and [56] for the compensation of input delays in multi-input linear systems.

Besides the unavailability of a systematic predictor-feedback design methodology for multi-input nonlinear systems with long input delays, the real motivation for this article comes from applications. Such systems serve as models for the dynamics of traffic [22], [49], teleoperators [24] and robotic manipulators [2], [20], motors [34], [52], multiagent systems [1], [18], [41], autonomous ground vehicles [40], unmanned aerial vehicles [23] and planar vertical take-off and landing aircrafts [21], [50], and the human musculoskeletal system in applications such as neuromuscular electrical stimulation [32], [39], [53], to name only a few. Motivated by the negative effects of input delays on the stability and performance of such control systems, in this article we present control designs that achieve delay compensation.

B. Contributions

We introduce a predictor-feedback control design for the compensation of long input delays in multi-input nonlinear systems. Since each individual input channel might induce a different delay the predictors of the plant's state are constructed recursively starting from the predictor that corresponds to the smallest input delay all the way through to the predictor that corresponds to the largest input delay. Specifically, at each step, the predictor, over the prediction horizon that corresponds to the current's step input delay, is constructed by actually predicting, over the appropriate prediction window, the future values of the predictor constructed at the previous step.

We conduct the stability analysis of the closed-loop system, under the developed predictor-feedback control law, utilizing two different techniques—one based on the construction of a Lyapunov functional and one based on estimates on the solutions of the closed-loop system. In the former case, the construction of a Lyapunov functional is enabled by the introduction of novel backstepping transformations of the actuator states, which are based on an equivalent, PDE representation of the constructed predictor states. In the latter approach, we exploit the fact that each delay is compensated after a finite time.

We present a detailed example, including numerical simulations, dealing with the stabilization of a nonholonomic robot subject to different input delays, in order to highlight the intricacies of our design and analysis methodologies. We specialize our results to the case of linear systems for which the predictor-feedback control laws are obtained explicitly and for which global exponential stability is achievable.

C. Organization

We start in Section II with the introduction to the problem of predictor-feedback stabilization of multi-input nonlinear systems and develop the predictor-feedback control laws. In Section III, we prove global asymptotic stability of the closed-loop system under predictor-feedback by constructing a Lyapunov functional and in Section IV we prove global asymptotic stability using estimates on solutions. Section V is devoted to a detailed example of stabilization of the nonholonomic unicycle subject to input delays. We specialize our methodology

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N. Bekiaris-Liberis is with the Department of Production Engineering and Management, Technical University of Crete, Chania 73100, Greece (e-mail: nikos.bekiaris@gmail.com).

M. Krstic is with the Department of Mechanical and Aerospace Engineering, University of California, San Diego, CA 92093 USA (e-mail: krstic@ucsd.edu).

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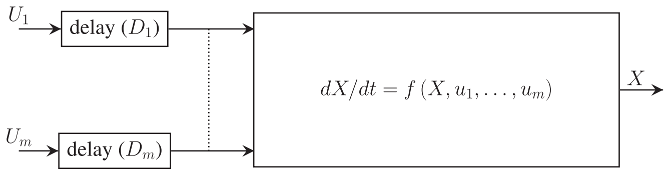

Fig. 1. Multi-input nonlinear system with distinct input delays.

to the case of linear systems in Section VI. For the example worked out in detail in Section V we present simulation results in Section VII. In Appendix A and B, we provide proofs of technical lemmas and results.

Notation: We use the common definition of class $\mathcal{K}$, $\mathcal{K}_\infty$ and $\mathcal{KL}$ functions from [33]. For an n-vector, the norm $|\cdot|$ denotes the usual Euclidean norm. For a function $u:[0,D]\times\mathbb{R}_+\to\mathbb{R}$ we denote by $\|u(\cdot,t)\|_\infty$ its spatial supremum norm, i.e., $\|u(\cdot,t)\|_\infty=\sup_{x\in[0,D]}|u(x,t)|$. For any $c>0$, we denote the spatially weighted supremum norm of u by $\|u(\cdot,t)\|_{c,\infty}=\sup_{x\in[0,D]}|e^{cx}u(x,t)|$. For a vector valued function $p:[0,D]\times\mathbb{R}_+\to\mathbb{R}^n$ we use a spatial supremum norm $\|p(\cdot,t)\|_\infty=\sup_{x\in[0,D]}\sqrt{p_1(x,t)^2+\ldots p_n(x,t)^2}$. We denote by $C^j(A;E)$ the space of functions that take values in E and have continuous derivatives of order j on A.

II. Multi-Input Nonlinear Systems With Distinct Delays and Predictor-Feedback Control Design

We consider the following system (see Fig. 1):

$$\dot{X}(t)=f\left(X(t),U_1(t-D_1),\ldots,U_m(t-D_m)\right) \quad (1)$$

where $X\in\mathbb{R}^n$ is state, $t\geq 0$ is time, $U_1,\ldots,U_m\in\mathbb{R}$ with initial conditions $U_{i_0}\in C[-D_i,0]$, $i=1,\ldots,m$, are control inputs, $D_1,\ldots,D_m$ are (potentially distinct) input delays satisfying (without loss of generality) $0<D_1\leq\cdots\leq D_m$, and $f:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^n$ is a locally Lipschitz vector field that satisfies $f(0,0,\ldots,0)=0$. The predictor feedback controllers are defined by

$$U_i(t)=\kappa_i\left(P_i(t)\right),\quad i=1,\ldots,m \quad (2)$$

where $\kappa_i:\mathbb{R}^n\to\mathbb{R}$, $i=1,\ldots,m$, are continuously differentiable feedback laws with $\kappa_i(0)=0$, $i=1,\ldots,m$, and, P_i are the D_i-time units ahead predictors of X, for all $i=1,\ldots,m$. Defining $D_{j,i}=D_j-D_i$, for all $i\leq j\leq m$, the predictors are given by

$$\begin{aligned}P_1(t)=X(t)+\int_{t-D_1}^{t}&f\left(P_1(\theta),U_1(\theta),U_2(\theta-D_{2,1})\right.\\&\left.\ldots,U_m(\theta-D_{m,1})\right)d\theta \quad (3)\\P_2(t)=P_1(t)+\int_{t-D_{2,1}}^{t}&f\left(P_2(\theta),\kappa_1\left(P_2(\theta)\right),U_2(\theta)\right.\\&\left.U_3(\theta-D_{3,2}),\ldots,U_m(\theta-D_{m,2})\right)d\theta \quad (4)\\&\vdots\\P_m(t)=P_{m-1}(t)+\int_{t-D_{m,m-1}}^{t}&f\left(P_m(\theta),\kappa_1\left(P_m(\theta)\right)\right.\\&\left.\kappa_2\left(P_m(\theta)\right),\ldots,\kappa_{m-1}\left(P_m(\theta)\right),U_m(\theta)\right)d\theta \quad (5)\end{aligned}$$

with initial conditions for the integral (3)–(5)

$$\begin{aligned}P_1(\theta)=X(0)+\int_{-D_1}^{\theta}&f\left(P_1(s),U_1(s),U_2(s-D_{2,1})\right.\\&\left.\ldots,U_m(s-D_{m,1})\right)ds,\quad -D_1\leq\theta\leq 0 \quad (6)\\P_2(\theta)=P_1(0)+\int_{-D_{2,1}}^{\theta}&f\left(P_2(s),\kappa_1\left(P_2(s)\right),U_2(s)\right.\\&\left.U_3(s-D_{3,2}),\ldots,U_m(s-D_{m,2})\right)ds\\&-D_{2,1}\leq\theta\leq 0 \quad (7)\\&\vdots\\P_m(\theta)=P_{m-1}(0)+\int_{-D_{m,m-1}}^{\theta}&f\left(P_m(s),\kappa_1\left(P_m(s)\right)\right.\\&\left.\kappa_2\left(P_m(s)\right),\ldots,\kappa_{m-1}\left(P_m(s)\right),U_m(s)\right)ds\\&-D_{m,m-1}\leq\theta\leq 0. \quad (8)\end{aligned}$$

We show that P_i, for all $i=1,\ldots,m$, are the D_i-time units ahead predictors of X by induction. In order to better understand the general induction step, we provide two initial steps. We show first that P_1 and P_2 are the D_1- and D_2-time units ahead predictors of X, respectively.

Step 1: We perform the change of variables $t=\theta+D_1$, for all $\theta\geq -D_1$, in (1) and define $P_1(\theta)=X(\theta+D_1)$, $\theta\geq -D_1$, to arrive at

$$\frac{dP_1(\theta)}{d\theta}=f\left(P_1(\theta),U_1(\theta),U_2(\theta-D_{2,1}),\ldots,U_m(\theta-D_{m,1})\right)\quad\text{for all }\theta\geq -D_1. \quad (9)$$

Integrating (9) from $\theta=t-D_1$ to $\theta=t$ and using definition $P_1(\theta)=X(\theta+D_1)$ we get (3). Integrating (9) from $\theta=-D_1$ to any $\theta\leq 0$ and using definition $P_1(-D_1)=X(0)$ we get (6).

Step 2: Performing the change of variables $\theta=s+D_{2,1}$, for all $s\geq -D_{2,1}$, in (9) and defining $P_2(s)=P_1(s+D_{2,1})=X(s+D_2)$, for all $s\geq -D_{2,1}$, we get that

$$\begin{aligned}\frac{dP_2(s)}{ds}=&f\left(P_2(s),\kappa_1\left(P_2(s)\right),U_2(s),U_3(s-D_{3,2})\right.\\&\left.\ldots,U_m(s-D_{m,2})\right),\quad\text{for all }s\geq -D_{2,1} \quad (10)\end{aligned}$$

where we also used the fact that $U_1(s+D_{2,1})=\kappa_1(P_1(s+D_{2,1}))$, for all $s+D_{2,1}\geq 0$, and definition $P_2(s)=P_1(s+D_{2,1})$. Integrating (10) from $s=t-D_{2,1}$ to $s=t$ and using definition $P_2(s)=P_1(s+D_{2,1})$, for all $s\geq -D_{2,1}$, we arrive at (4). Integrating (10) from $\theta=-D_{2,1}$ to any $\theta\leq 0$ and using definition $P_2(-D_{2,1})=P_1(0)$ we get (7).

Step j: Assume now that the D_j-time units ahead predictor of X, namely P_j, satisfies the following ODE in r:

$$\begin{aligned}\frac{dP_j(r)}{dr}=&f\left(P_j(s),\kappa_1\left(P_j(r)\right),\ldots,\kappa_{j-1}\left(P_j(r)\right),U_j(r)\right.\\&\left.U_{j+1}(r-D_{j+1,j}),\ldots,U_m(r-D_{m,j})\right)\\&\text{for all }r\geq -D_{j,j-1}. \quad (11)\end{aligned}$$

Performing the change of variables $r = h + D_{j+1,j}$, for all $h \geq -D_{j+1,j}$, in (11) and defining $P_{j+1}(h) = P_j(h + D_{j+1,j}) = X(h + D_{j+1})$, for all $h \geq -D_{j+1,j}$, we get that

$$\frac{dP_{j+1}(h)}{dh} = f\left(P_{j+1}(h), \kappa_1\left(P_{j+1}(h)\right), \kappa_j\left(P_{j+1}(h)\right)\right. \\ \left. U_{j+1}(h), \ldots, U_m(h - D_{m,j+1})\right) \\ \text{for all } h \geq -D_{j+1,j} \tag{12}$$

where we also used the fact that $U_j(h + D_{j+1j,}) = \kappa_j(P_j(h + D_{j+1,j}))$, for all $h + D_{j+1,j} \geq 0$, and definition $P_{j+1}(h) = P_j(h + D_{j+1,j})$. Integrating (12) from $h = t - D_{j+1,j}$ to $h = t$ and from $h = -D_{j+1j}$ to any $h \leq 0$, and using definition $P_{j+1}(h) = P_j(h + D_{j+1,j})$, for all $h \geq -D_{j+1,j}$ (which implies that $P_{j+1}(-D_{j+1,j}) = P_j(0)$), we conclude that indeed the D_i-time units ahead predictors of X, for all $i = 1, \ldots, m$, are given by (3)–(5) with initial conditions (6)–(8).

Although implementation and approximation issues are beyond the scope of this paper, which focuses on fundamental continuous-time designs, there is a large body of recent research that is concentrated almost exclusively on implementation and approximation issues for nonlinear predictor feedback laws, which includes [26], [27], [30] (see also, for instance, [48] and [58] for linear predictor feedback laws). In particular, it can be shown that both an appropriate approximation and an appropriate dynamic implementation (among different implementation possibilities) of the nonlinear predictor feedback law can be employed for the stabilization of the nonlinear delay system. Thus, one can in principle try to adapt the techniques developed in [26], [27], and [30] to systems with multiple, distinct input delays.

III. Lyapunov-Based Stability Analysis Under Predictor Feedback

Assumption 1: The system $\dot{X} = f(X, \omega_1, \ldots, \omega_m)$ is strongly forward complete with respect to $\omega = (\omega_1, \ldots, \omega_m)^T$.

Assumption 2: The system $\dot{X} = f(X, \omega_1 + \kappa_1(X), \ldots, \omega_m + \kappa_m(X))$ is input-to-state stable with respect to $\omega = (\omega_1, \ldots, \omega_m)^T$.

Assumption 3: The systems $\dot{X} = g_j(X, \omega_{j+1}, \ldots, \omega_m)$, for all $j = 1, \ldots, m-1$, with $g_j(X, \omega_{j+1}, \ldots, \omega_m) = f(X, \kappa_1(X), \ldots, \kappa_j(X), \omega_{j+1}, \ldots, \omega_m)$, are strongly forward complete with respect to $\omega = (\omega_{j+1}, \ldots, \omega_m)^T$.

The definitions of strong forward completeness and input-to-state stability are those from [37] (see also [3] for the definition of standard forward completeness which differs from strong forward completeness in that $f(0, 0, \ldots, 0) = 0$) and [54], respectively.

Assumption 1 guarantees that for every initial condition and every locally bounded input signal the corresponding solution is defined for all $t \geq 0$. In particular, the plant does not exhibit finite escape before the first feedback control reaches it. This is a natural requirement for achieving global stabilization in the presence of arbitrary large delays affecting the inputs of a system. Assumption 2 can be relaxed to only global asymptotic stability of system $\dot{X} = f(X, \kappa_1(X), \ldots, \kappa_m(X))$. Yet, at the expense of not having a Lyapunov functional available. Assumption 3 guarantees that after the j-th controller "kicks in" and the D_j-th delay is compensated, and hence, the plant behaves according to $\dot{X} = f(X, \kappa_1(X), \ldots, \kappa_j(X), U_{j+1}(t - D_{j+1}), \ldots, U_m(t - D_m))$, the solutions are also well-defined. In particular, the plant does not exhibit finite escape before the $j+1$-th feedback control reaches it and after the jth feedback control has already reached the plant. Note that Assumption 3 can be relaxed to strong forward completeness of systems $\dot{X} = g_j(X, \omega_{j+1}, \ldots, \omega_m)$ with respect to $\omega = (\omega_{j+1}, \ldots, \omega_m)^T$, for all $j \in \{r_1, r_1 + r_2, \ldots, r_1 + \cdots + r_\nu\}$, where $g_j(X, \omega_{j+1}, \ldots, \omega_m) = f(X, \kappa_1(X), \ldots, \kappa_j(X), \omega_{j+1}, \ldots, \omega_m)$, r_1 denotes the number of delays that are equal to D_1, r_σ, $\sigma = 2, \ldots, \nu$, denotes the number of delays that are equal to $D_{r_1 + \cdots + r_{\sigma-1}}$, and ν is the number of distinct delays. In particular, when all delays are identical, Assumption 3 can be completely removed.

The stability proof is based on an equivalent representation of plant (1), using transport PDEs for the actuator states, and on an equivalent PDE representation of the predictor states (3)–(5). We present the alternative representations for the plant and the predictor states before stating and proving the main result of this section, since the reader might find the alternative formalisms helpful in better digesting the design and analysis ideas of our methodology.

A. Equivalent Representation of the Plant Using Transport PDEs for the Actuator States

System (1) can be written equivalently as

$$\dot{X}(t) = f\left(X(t), u_1(0, t), \ldots, u_m(0, t)\right) \tag{13}$$

$$\partial_t u_i(x, t) = \partial_x u_i(x, t), \ x \in (0, D_i), \quad i = 1, \ldots, m \tag{14}$$

$$u_i(D_i, t) = U_i(t), \quad i = 1, \ldots, m. \tag{15}$$

To see this note that the solutions to (14) and (15) are given by

$$u_i(x, t) = U_i(t + x - D_i), \quad x \in [0, D_i], \quad i = 1, \ldots, m. \tag{16}$$

B. Transport PDE Representation of the Predictor States

The predictor states $P_1(\theta)$, for all $\theta \geq -D_1$, and $P_j(\theta)$, for all $\theta \geq -D_{j,j-1}$ and $j = 2, \ldots, m$, can be written equivalently as (see Fig. 2)

$$p_1(x, t) = X(t) + \int_0^x f\left(p_1(y, t), u_1(y, t)\right. \\ \left. \ldots, u_m(y, t)\right) dy, \quad x \in [0, D_1] \tag{17}$$

$$p_2(x, t) = p_1(D_1, t) + \int_{D_1}^x f\left(p_2(y, t), \kappa_1\left(p_2(y, t)\right), u_2(y, t)\right. \\ \left. \ldots, u_m(y, t)\right) dy, \quad x \in [D_1, D_2] \tag{18}$$

$$\vdots$$

$$p_m(x, t) = p_{m-1}(D_{m-1}, t) + \int_{D_{m-1}}^x f\left(p_m(y, t)\right. \\ \kappa_1\left(p_m(y, t)\right), \ldots, \kappa_{m-1}\left(p_m(y, t)\right) \\ \left. u_m(y, t)\right) dy, \quad x \in [D_{m-1}, D_m]. \tag{19}$$

We show this by induction. In order to make the presentation of the procedure clearer, we present two steps before the general step. We first observe that (see Section II) $P_1(\theta) = X(\theta + D_1)$, for all $\theta \geq -D_1$, and $P_j(\theta) = X(\theta + D_j)$, for all $\theta \geq -D_{j,j-1}$

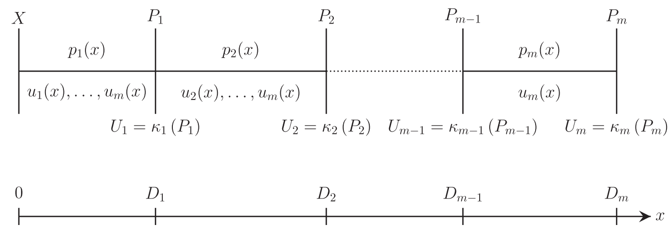

Fig. 2. The D_j-time units ahead predictors of the state X, namely P_j, given in (3)–(5), and their equivalent representation by the PDE states p_j, given in (17)–(19), based on the transport-PDE equivalent of the actuator states defined in (14) and (15). The control laws U_j are defined in (2) in terms of P_j and can be written equivalently as in (42) in terms of p_j.

and $j=2,\dots,m$. The functions p_i, $i=1,\dots,m$, satisfy the following ODEs in x:

$$\begin{aligned}\partial_x p_1(x,t) &= f\left(p_1(x,t),u_1(x,t),\dots,u_m(x,t)\right)\\ &\quad x\in[0,D_1] \qquad (20)\\ \partial_x p_2(x,t) &= f\left(p_2(x,t),\kappa_1\left(p_2(x,t)\right),u_2(x,t)\right.\\ &\quad \left.\dots,u_m(x,t)\right),\quad x\in[D_1,D_2] \qquad (21)\\ &\vdots\\ \partial_x p_m(x,t) &= f\left(p_m(x,t),\kappa_1\left(p_m(x,t)\right),\dots\right.\\ &\quad \left.\kappa_{m-1}\left(p_m(x,t)\right),u_m(x,t)\right)\\ &\quad x\in[D_{m-1},D_m] \qquad (22)\end{aligned}$$

with initial conditions

$$\begin{aligned}p_1(0,t) &= X(t) \qquad (23)\\ p_2(D_1,t) &= p_1(D_1,t) \qquad (24)\\ &\vdots\\ p_m(D_{m-1},t) &= p_{m-1}(D_{m-1},t). \qquad (25)\end{aligned}$$

Step 1: The solution to (20) and (23) is

$$p_1(x,t)=X(t+x),\quad x\in[0,D_1]. \qquad (26)$$

In order to show this, first note that (26) satisfies the boundary condition (23). The function $X(t+x)$ also satisfies the ODE in x (20) which follows from the fact that by (1) one can conclude that for all $t\geq 0$ and $0\leq x\leq D_1$:

$$X'(t+x)=f\left(X(t+x),U_1(t+x-D_1),\right.\\ \left.\dots,U_m(t+x-D_m)\right). \qquad (27)$$

The result follows from the uniqueness of solutions to the ODE (1). Therefore, by defining

$$p_1(D_1,t)=P_1(t) \qquad (28)$$

and using the fact that p_1 is a function of one variable, namely $x+t$ [which follows from (26)], one can conclude that:

$$P_1(t+x-D_1)=p_1(x,t),\quad x\in[0,D_1]. \qquad (29)$$

Performing the change of variables $x=\theta+D_1-t$, for all $t-D_1\leq\theta\leq t$, in (17) and using (16), (23), and (29) we arrive at

$$P_1(\theta)=X(t)+\int_{t-D_1}^{\theta} f\left(P_1(s),U_1(s),\dots,U_m(s-D_{m,1})\right)ds\\ t-D_1\leq\theta\leq t. \qquad (30)$$

Step 2: Similarly, it can be shown that

$$p_2(x,t)=X(t+x),\quad x\in[D_1,D_2] \qquad (31)$$

is the solution to (21) and (24) since it satisfies (24) and since it satisfies the ODE in x (21) which follows from the fact that the function $X(t+x)$ satisfies:

$$X'(t+x)=f\left(X(t+x),\kappa_1\left(X(t+x)\right)\right.\\ \left.U_2(t+x-D_2),\dots,U_m(t+x-D_m)\right)\\ \text{for all } t\geq 0 \text{ and } D_1\leq x\leq D_2 \qquad (32)$$

where we also used the fact that $U_1(t+x-D_1)=\kappa_1(P_1(t+x-D_1))=\kappa_1(X(t+x))$, for all $x\geq D_1$. Defining

$$p_2(D_2,t)=P_2(t) \qquad (33)$$

and using the fact that p_2 is a function of one variable, namely $x+t$ [which follows from (31)], one can conclude that:

$$P_2(t+x-D_2)=p_2(x,t),\quad x\in[D_1,D_2]. \qquad (34)$$

Performing the change of variables $x=\theta+D_2-t$, for all $t-D_{21}\leq\theta\leq t$, in (18) and using (16), (28), and (34) we arrive at

$$P_2(\theta)=P_1(t)+\int_{t-D_{2,1}}^{\theta} f\left(P_2(s),\kappa_1\left(P_2(s)\right),U_2(s)\right.\\ \left.U_3(s-D_{3,2}),\dots,U_m(s-D_{m,2})\right)ds\\ \text{for all } t-D_{2,1}\leq\theta\leq t. \qquad (35)$$

Step j: In general, assume that for some j

$$\begin{aligned}p_j(x,t) &= P_j(t+x-D_j)\\ &= X(t+x),\quad x\in[D_{j-1},D_j]. \qquad (36)\end{aligned}$$

We show next that $p_{j+1}(x,t)=X(t+x)$, for all $x\in[D_j,D_{j+1}]$. We first observe that $p_{j+1}(D_j,t)=X(t+D_j)=p_j(D_j,t)$. Moreover, the function $p_{j+1}(x,t)=X(t+x)$, for all $x\in[D_j,D_{j+1}]$, satisfies the following ODE in x:

$$\partial_x p_{j+1}(x,t)=f\left(p_{j+1}(x,t),\kappa_1\left(p_{j+1}(x,t)\right),\dots\right.\\ \kappa_j\left(p_{j+1}(x,t)\right),u_{j+1}(x,t)\\ \left.\dots,u_m(x,t)\right),\quad x\in[D_j,D_{j+1}] \qquad (37)$$

since the following holds for all $t\geq 0$:

$$X'(t+x)=f\left(X(t+x),\kappa_1\left(X(t+x)\right),\dots\right.\\ \kappa_j\left(X(t+x)\right),U_{j+1}(t+x-D_{j+1}),\dots\\ \left.U_m(t+x-D_m)\right),\quad D_j\leq x\leq D_{j+1} \qquad (38)$$

where we also used the fact that $U_i(t+x-D_i)=P_i(t+x-D_i)=X(t+x)$, for all $x\geq D_j$ and $i\leq j$. By defining

$$p_{j+1}(D_{j+1},t)=P_{j+1}(t) \qquad (39)$$

once can conclude that

$$P_{j+1}(t+x-D_{j+1})=p_{j+1}(x,t),\quad x\in[D_j,D_{j+1}]. \qquad (40)$$

Performing the change of variables $x = \theta + D_{j+1} - t$, for all $t - D_{j+1} \leq \theta \leq t$, we arrive at

$$P_{j+1}(\theta) = P_j(t) + \int_{t-D_{j+1,j}}^{\theta} f\left(P_{j+1}(s), \kappa_1\left(P_{j+1}(s)\right)\right. \\ \ldots, \kappa_j\left(P_j(s)\right), U_{j+1}(s), U_{j+2}(s - D_{j+2,j+1}) \\ \left.\ldots, U_m(s - D_{m,j+1})\right) ds \tag{41}$$

for all $t - D_{j,j+1} \leq \theta \leq t$ which completes the proof.

Note that with this representation we have that

$$U_i(t) = \kappa_i\left(p_i(D_i, t)\right), \quad i = 1, \ldots, m. \tag{42}$$

C. Main Result and Its Proof

Theorem 1: Consider the closed-loop system consisting of the plant (13)–(15) and the control laws (42), (17)–(19). Under Assumptions 1, 2, and 3, there exists a class $\mathcal{KL}$ function β such that for all initial conditions $X_0 \in \mathbb{R}^n$ and $u_{i_0} \in C[0, D_i]$, $i = 1, \ldots, m$, which are compatible with the feedback laws, that is, they satisfy $u_{i_0}(D_i) = \kappa_i(p_i(D_i, 0))$, $i = 1, \ldots, m$, the closed-loop system has a unique solution $X(t) \in C^1[0, \infty)$ and $u_i(x, t) \in C([0, D_i] \times [0, \infty))$, $i = 1, \ldots, m$, and the following holds:

$$\Xi(t) \leq \beta\left(\Xi(0), t\right), \quad \text{for all } t \geq 0 \tag{43}$$

where

$$\Xi(t) = |X(t)| + \sum_{i=1}^{m} \|u_i(\cdot, t)\|_\infty. \tag{44}$$

Corollary 1 (Theorem 1 in Standard Delay Notation): Consider the closed-loop system consisting of the plant (1) and the control laws (2)–(8). Under Assumptions 1, 2, and 3, the following holds:

$$\Omega(t) \leq \beta\left(\Omega(0), t\right), \quad \text{for all } t \geq 0 \tag{45}$$

where

$$\Omega(t) = |X(t)| + \sum_{i=1}^{m} \sup_{t-D_i \leq \theta \leq t} |U_i(\theta)|. \tag{46}$$

The proof of Theorem 1 is based on a series of technical lemmas which are presented next and whose proofs are provided in Appendix A. Corollary 1 follows immediately from Theorem 1 by using (16).

Lemma 1: The backstepping transformations of u_i, $i = 1, \ldots, m$, defined by

$$w_1(x, t) = u_1(x, t) - \kappa_1\left(p_1(x, t)\right), \quad x \in [0, D_1] \tag{47}$$

$$w_2(x, t) = u_2(x, t) - \begin{cases} \kappa_2\left(p_1(x, t)\right), & x \in [0, D_1] \\ \kappa_2\left(p_2(x, t)\right), & x \in [D_1, D_2] \end{cases} \tag{48}$$

$$\vdots$$

$$w_m(x, t) = u_m(x, t) - \begin{cases} \kappa_m\left(p_1(x, t)\right), & x \in [0, D_1] \\ \kappa_m\left(p_2(x, t)\right), & x \in [D_1, D_2] \\ \vdots & \\ \kappa_m\left(p_m(x, t)\right), & x \in [D_{m-1}, D_m] \end{cases} \tag{49}$$

where p_i, $i = 1, \ldots, m$, are defined in (17)–(19), together with the control laws (42), (17)–(19), transform system (13)–(15) to the following "target system":

$$\dot{X}(t) = f\left(X(t), w_1(0, t) + \kappa_1\left(X(t)\right), \ldots, w_m(0, t)\right. \\ \left. + \kappa_m\left(X(t)\right)\right) \tag{50}$$

$$\partial_t w_i(x, t) = \partial_x w_i(x, t), \quad x \in (0, D_i), \quad i = 1, \ldots, m \tag{51}$$

$$w_i(D_i, t) = 0, \quad i = 1, \ldots, m. \tag{52}$$

Lemma 2: The inverse backstepping transformations of (47)–(49) are defined by

$$u_1(x, t) = w_1(x, t) + \kappa_1\left(\pi_1(x, t)\right), \quad x \in [0, D_1] \tag{53}$$

$$u_2(x, t) = w_2(x, t) + \begin{cases} \kappa_2\left(\pi_1(x, t)\right), & x \in [0, D_1] \\ \kappa_2\left(\pi_2(x, t)\right), & x \in [D_1, D_2] \end{cases} \tag{54}$$

$$\vdots$$

$$u_m(x, t) = w_m(x, t) + \begin{cases} \kappa_m\left(\pi_1(x, t)\right), & x \in [0, D_1] \\ \kappa_m\left(\pi_2(x, t)\right), & x \in [D_1, D_2] \\ \vdots & \\ \kappa_m\left(\pi_m(x, t)\right), & x \in [D_{m-1}, D_m] \end{cases} \tag{55}$$

where

$$\pi_1(x, t) = X(t) + \int_0^x f\left(\pi_1(y, t), w_1(y, t) + \kappa_1\left(\pi_1(y, t)\right)\right. \\ \left. \ldots, w_m(y, t) + \kappa_m\left(\pi_1(y, t)\right)\right) dy \\ x \in [0, D_1] \tag{56}$$

$$\pi_2(x, t) = \pi_1(D_1, t) + \int_{D_1}^x f\left(\pi_2(y, t), \kappa_1\left(\pi_2(y, t)\right)\right. \\ w_2(y, t) + \kappa_2\left(\pi_2(y, t)\right), \ldots, w_m(y, t) \\ \left. + \kappa_m\left(\pi_1(y, t)\right)\right) dy, \quad x \in [D_1, D_2] \tag{57}$$

$$\vdots$$

$$\pi_m(x, t) = \pi_{m-1}(D_{m-1}, t) + \int_{D_{m-1}}^x f\left(\pi_m(y, t)\right. \\ \kappa_1\left(\pi_m(y, t)\right), \ldots, \kappa_{m-1}\left(\pi_m(y, t)\right) \\ \left. w_m(y, t) + \kappa_m\left(\pi_m(y, t)\right)\right) dy \\ x \in [D_{m-1}, D_m]. \tag{58}$$

Lemma 3: There exists a class $\mathcal{KL}$ function β_1 such that the following holds:

$$\bar{\Xi}(t) \leq \beta_1\left(\bar{\Xi}(0), t\right), \quad \text{for all } t \geq 0 \tag{59}$$

where

$$\bar{\Xi}(t) = |X(t)| + \sum_{i=1}^{m} \|w_i(\cdot, t)\|_\infty. \tag{60}$$

Lemma 4: There exist class $\mathcal{K}_\infty$ functions $\rho_1, \ldots, \rho_m$ such that

$$\|p_1(\cdot,t)\|_\infty \le \rho_1\left(\Xi(t)\right) \quad (61)$$
$$\|p_2(\cdot,t)\|_\infty \le \rho_2\left(\Xi(t)\right) \quad (62)$$
$$\vdots$$
$$\|p_m(\cdot,t)\|_\infty \le \rho_m\left(\Xi(t)\right) \quad (63)$$

where Ξ is defined in (44).

Lemma 5: There exist class $\mathcal{K}_\infty$ functions $\bar\rho_1, \ldots, \bar\rho_m$ such that

$$\|\pi_1(\cdot,t)\|_\infty \le \bar\rho_1\left(\bar\Xi(t)\right) \quad (64)$$
$$\|\pi_2(\cdot,t)\|_\infty \le \bar\rho_2\left(\bar\Xi(t)\right) \quad (65)$$
$$\vdots$$
$$\|\pi_m(\cdot,t)\|_\infty \le \bar\rho_m\left(\bar\Xi(t)\right) \quad (66)$$

where $\bar\Xi$ is defined in (60).

Lemma 6: There exist class $\mathcal{K}_\infty$ functions ρ, $\bar\rho$ such that

$$\bar\Xi(t) \le \rho\left(\Xi(t)\right) \quad (67)$$
$$\Xi(t) \le \bar\rho\left(\bar\Xi(t)\right). \quad (68)$$

Proof of Theorem 1: Combining (68) with (59) we get that $\Xi(t) \le \bar\rho(\beta_1(\bar\Xi(0),t))$, for all $t \ge 0$, and hence, with (67) we arrive at (43) with $\beta(s,t) = \bar\rho(\beta_1(\rho(s),t))$. The proof of existence and uniqueness of a solution $X(t) \in C^1[0,\infty)$ and $u_i(x,t) \in C([0,D_i]\times[0,\infty))$, $i = 1,\ldots,m$, is shown as follows. Using relations (20)–(22) for $t = 0$, the compatibility of the initial conditions u_{i_0}, $i = 1,\ldots,m$, with the feedback laws (42) guarantee that $p_i(x,0) \in C^1[D_{i-1},D_i]$, where $D_0 = 0$. Hence, using relations (17)–(19), (23)–(25), and the fact that $u_{i_0} \in C[0,D_i]$, $i = 1,\ldots,m$, it also follows from (47)–(49) that $w_{i_0} \in C[0,D_i]$, $i = 1,\ldots,m$. The solutions to (51) and (52) are given for all $i = 1,\ldots,m$ by

$$w_i(x,t) = \begin{cases} w_{i_0}(t+x), & 0 \le x+t < D_i \\ 0, & x+t \ge D_i. \end{cases} \quad (69)$$

The uniqueness of this solution follows from the uniqueness of the solution to (51) and (52) (see Sections 2.1 and 2.3 in [19]). Hence, the compatibility of the initial conditions u_{i_0}, $i = 1, \ldots, m$, with the feedback laws (42) guarantee that there exist a unique solution $w_i(x,t) \in C([0,D_i]\times[0,\infty))$, $i = 1,\ldots,m$. From the target system (50) it follows that $X(t) \in C^1[0,\infty)$. The fact that $\pi_i(x,t) = X(t+x)$, for all $t \ge 0$ and $x \in [D_{i-1},D_i]$, and the inverse backstepping transformations (53)–(55) guarantee that $u_i(x,t) \in C([0,D_i]\times[0,\infty))$, $i = 1,\ldots,m$. The proof is completed.

IV. Stability Analysis Under Predictor Feedback Using Estimates on Solutions

Theorem 2: Consider the closed-loop system consisting of the plant (1) and the control laws (2)–(4). Let Assumptions 1 and 3 hold and assume that the system $\dot X = f(X,\kappa_1(X), \ldots, \kappa_m(X))$ is globally asymptotically stable. There exists a class $\mathcal{KL}$ function $\hat\beta$ such that for all initial conditions $X_0 \in \mathbb{R}^n$ and $U_{i_0} \in C[-D_i,0]$, $i = 1,\ldots,m$, which are compatible with the feedback laws, that is, they satisfy $U_{i_0}(0) = \kappa_i(P_i(0))$, $i = 1,\ldots,m$, there exists a unique solution to the closed-loop system with $X(t) \in C^1[0,\infty)$ and $U_i(t)$, $i = 1,\ldots,m$, locally Lipschitz on $(0,\infty)$, and the following holds:

$$\Omega(t) \le \hat\beta\left(\Omega(0),t\right), \quad \text{for all } t \ge 0 \quad (70)$$

where Ω is defined in (46).

Proof: We estimate first $|X(t)|$, for all $t \ge 0$. Since the system $\dot X = f(X,\omega_1,\ldots,\omega_m)$ is forward complete using Lemma 3.5 from [25] and the fact that $f(0,0,\ldots,0) = 0$ (which allows us to set $R = 0$), we get that

$$|X(t)| \le \psi_1\left(\Omega(0)\right), \quad \text{for all } 0 \le t \le D_1 \quad (71)$$

for some class $\mathcal{K}_\infty$ function ψ_1. Using the fact that $U_1(t) = \kappa_1(P_1(t))$, for all $t \ge 0$, with $P_1(t) = X(t+D_1)$ and the fact that system $\dot X = f(X,\kappa_1(X),\omega_2,\ldots,\omega_m)$ is forward complete with respect to ω_2 we get by applying Lemma 3.5 from [25] that $|X(t)| \le \bar\psi_2(|X(D_1)| + \sum_{i=2}^m \sup_{D_1-D_i\le\theta\le D_2-D_i} |U_i(\theta)|)$, for all $D_1 \le t \le D_2$, for some class $\mathcal{K}_\infty$ function $\bar\psi_2$. Hence, since $D_j \le D_i$, $\forall\, j \le i$, with (71) we get

$$|X(t)| \le \psi_2\left(\Omega(0)\right), \quad \text{for all } D_1 \le t \le D_2 \quad (72)$$

where the class $\mathcal{K}_\infty$ function $\psi_2(s)$ is defined as $\psi_2(s) = \bar\psi_2(\psi_1(s)+s)$. By repeatedly applying Lemma 3.5 from [25] we get under Assumption 3 that there exist class $\mathcal{K}_\infty$ functions $\bar\psi_j$, $j = 3,\ldots,m$ such that $|X(t)| \le \bar\psi_j(|X(D_{j-1})| + \sum_{i=j}^m \sup_{D_{j-1}-D_i\le\theta\le D_j-D_i} |U_i(\theta)|)$, for all $D_{j-1} \le t \le D_j$, and hence

$$|X(t)| \le \psi_j\left(\Omega(0)\right), \quad \text{for all } D_{j-1} \le t \le D_j \quad (73)$$

where the class $\mathcal{K}_\infty$ functions ψ_j, $j = 3,\ldots,m$ are defined as $\psi_j(s) = \bar\psi_j(\psi_{j-1}(s)+s)$, $j = 3,\ldots,m$, and where we also used the fact that $D_j \le D_i$, $\forall\, j \le i$. Combining (71)–(73) we arrive at

$$|X(t)| \le \psi\left(\Omega(0)\right), \quad \text{for all } 0 \le t \le D_m \quad (74)$$

where $\psi(s) = \sum_{i=1}^m \psi_i(s)$. Using the fact that $U_m(t) = \kappa_m(P_m(t))$, for all $t \ge 0$, with $P_m(t) = X(t+D_m)$ and the fact that system $\dot X = f(X,\kappa_1(X),\ldots,\kappa_m(X))$ is globally asymptotically stable we get that $|X(t)| \le \hat\beta_1(|X(D_m)|, t - D_m)$, for all $t \ge D_m$, for some class $\mathcal{KL}$ function $\hat\beta_1$. Hence, using (74) we get that

$$|X(t)| \le \hat\beta_2\left(\Omega(0),t\right), \quad \text{for all } t \ge 0 \quad (75)$$

where the class $\mathcal{KL}$ function $\hat\beta_2$ is given by $\hat\beta_2(s,t) = \hat\beta_1(\psi(s),\max\{0,t-D_m\}) + \psi(s)e^{-\max\{0,t-D_m\}}$.

We estimate next $\sup_{t-D_1\le\theta\le t} |U_1(\theta)|$. Since κ_1 is locally Lipschitz and $\kappa_1(0) = 0$, there exists a class $\mathcal{K}_\infty$ function $\hat\alpha_1$ such that $|\kappa_1(X)| \le \hat\alpha_1(|X|)$, for all $X \in \mathbb{R}^n$. Since for all $t \ge 0$ it holds that $U_1(t) = \kappa_1(X(t+D_1))$ using (75) one can conclude that

$$\sup_{t-D_1\le\theta\le t} |U_1(\theta)| \le \sup_{-D_1\le\theta\le 0} |U_1(\theta)| + \hat\alpha_1\left(\hat\beta_2\left(\Omega(0),0\right)\right)$$
$$0 \le t \le D_1 \quad (76)$$

and hence

$$\sup_{t-D_1\le\theta\le t}|U_1(\theta)| \le \hat{\alpha}_2(\Omega(0)), \quad 0\le t\le D_1 \tag{77}$$

where the function $\hat{\alpha}_2(s)=\hat{\alpha}_1(\hat{\beta}_2(s,0))+s$ belongs to class $\mathcal{K}_\infty$. Using identical arguments we also get that

$$\sup_{t-D_1\le\theta\le t}|U_1(\theta)| \le \hat{\alpha}_1\left(\hat{\beta}_2(\Omega(0),t)\right), \quad t\ge D_1. \tag{78}$$

Combining (77) with (78) we arrive at

$$\sup_{t-D_1\le\theta\le t}|U_1(\theta)| \le \hat{\beta}_3(\Omega(0),t), \quad \text{for all } t\ge 0 \tag{79}$$

where the class $\mathcal{KL}$ function $\hat{\beta}_3$ is defined by $\hat{\beta}_3(s,t)=\hat{\alpha}_1(\hat{\beta}_2(\Omega(0),\max\{0,t-D_1\}))+\hat{\alpha}_2(s)\ \ e^{-\max\{0,t-D_1\}}$. Using the facts that κ_2 is locally Lipschitz and that $\kappa_2(0)=0$, which allows one to conclude that there exists a class $\mathcal{K}_\infty$ function $\hat{\alpha}_3$ such that $|\kappa_2(X)|\le\hat{\alpha}_3(|X|)$, for all $X\in\mathbb{R}^n$, with similar arguments we get that

$$\sup_{t-D_2\le\theta\le t}|U_2(\theta)| \le \hat{\beta}_4(\Omega(0),t), \quad \text{for all } t\ge 0 \tag{80}$$

where the class $\mathcal{KL}$ function $\hat{\beta}_4$ is defined by $\hat{\beta}_4(s,t)=\hat{\alpha}_3(\hat{\beta}_2(\Omega(0),\max\{0,t-D_2\}))+\hat{\alpha}_4(s)\ \ e^{-\max\{0,t-D_2\}}$, where the function $\hat{\alpha}_4(s)=\hat{\alpha}_3(\hat{\beta}_2(s,0))+s$ belongs to class $\mathcal{K}_\infty$. With the same arguments one can conclude that there exist class $\mathcal{KL}$ functions $\hat{\beta}_{j+2}$, $j=3,\ldots,m$, such that

$$\sup_{t-D_j\le\theta\le t}|U_j(\theta)| \le \hat{\beta}_{j+2}(\Omega(0),t), \quad j=3,\ldots,m, \quad \text{for all } t\ge 0. \tag{81}$$

Combining estimates (75), (79)–(81), we get (70) with $\hat{\beta}(s,t)=\hat{\beta}_2(s,t)+\sum_{i=1}^m\hat{\beta}_{i+2}(s,t)$. From (1) using the fact that $U_{i_0}\in C[-D_i,0]$, for all $i=1,\ldots,m$, the Lipschitzness of the vector field f guarantees that $X(t)\in C^1[0,D_1)$. The fact that $U_1(t-D_1)=\kappa_1(X(t))$, for all $t\ge D_1$, and the Lipschitzness of κ_1 guarantee that $X(t)\in C^1(D_1,D_2)$. Since U_{1_0} is compatible with the first feedback law one can conclude that $X(t)\in C^1[0,D_2)$. Analogously, since for $t\ge D_1$ the state X evolves according to $\dot{X}(t)=f(X(t),\kappa_1(X(t)),U_2(t-D_2),\ldots,U_m(t-D_m))$, the fact that U_{2_0} is compatible with the second feedback law and the fact that $U_2(t-D_2)=\kappa_2(X(t))$, for all $t\ge D_2$, where κ_2 is locally Lipschitz, guarantee that $X(t)\in C^1[0,D_3)$. Continuing this procedure it is shown that $X(t)\in C^1[0,\infty)$. Since $U_i(t)=\kappa_i(X(t+D_i))$, $i=1,\ldots,m$, the Lipshitzness of κ_i, $i=1,\ldots,m$, guarantees that $U_i(t)$, $i=1,\ldots,m$, are Lipschitz on $(0,\infty)$. ■

V. Stabilization of the Nonholonomic Unicycle Subject to Distinct Input Delays

We consider the following system:

$$\dot{X}_1(t)=U_2(t-D_2)\cos(X_3(t)) \tag{82}$$

$$\dot{X}_2(t)=U_2(t-D_2)\sin(X_3(t)) \tag{83}$$

$$\dot{X}_3(t)=U_1(t-D_1) \tag{84}$$

which describes the dynamics of the unicycle, where (X_1,X_2) is the position of the robot, X_3 is heading, U_2 is speed, and U_1 is the turning rate. We consider the following time-varying nominal control law designed in [51]:

$$U_1(t)=-M(t)^2\cos(t)-M(t)Q(t)\left(1+\cos^2(t)\right)-X_3(t) \tag{85}$$

$$U_2(t)=-M(t)+Q(t)(\sin(t)-\cos(t))+Q(t)U_1(t) \tag{86}$$

where

$$M(t)=X_1(t)\cos(X_3(t))+X_2(t)\sin(X_3(t)) \tag{87}$$

$$Q(t)=X_1(t)\sin(X_3(t))-X_2(t)\cos(X_3(t)) \tag{88}$$

which achieves global uniform asymptotic stabilization when $D_1=D_2=0$ [51]. We verify in Appendix B that Assumptions 1 and 3 are verified as well for system (82)–(84) under the control laws (85)–(88), and hence, one can apply, Theorem 2.

Note that when $D_1>D_2$, Assumption 3 may not hold. This fact can be seen as follows. It can be shown that $\dot{M}(t)=U_2(t-D_2)-Q(t)U_1(t-D_1)$, $\dot{Q}(t)=M(t)U_1(t-D_1)$, and $\dot{X}_3(t)=U_1(t-D_1)$. Therefore, when $D_1>D_2$ in order for Assumption 3 to hold, the system $\dot{Y}=f(Y,U_2(t,Y),\omega_2)$ with $f(Y,U_2(t,Y),\omega_2)=\begin{bmatrix}U_2(t,Y)-Y_2\omega_2\\ Y_1\omega_2\\ \omega_2\end{bmatrix}$, where $U_2(t,Y)=-Y_1+Y_2(\sin(t)-\cos(t))-Y_1^2Y_2\cos(t)-Y_1Y_2^2(1+\cos^2(t))-Y_2Y_3$, has to be forward complete with respect to ω_2. However, this might not be the case. Consider, for example, the case in which $D_2=0$, $\omega_2\equiv 0$, $Y_2(0)=1$, $Y_3(0)=0$, and hence, $Y_2(t)=1$, for all $t\ge 0$. The state Y_1 satisfies $\dot{Y}_1=-Y_1+\sin(t)-\cos(t)-Y_1^2\cos(t)-Y_1(1+\cos^2(t))$, for all $t\ge 0$. We show next that Y_1 escapes to infinity before $t=\pi/4$. Choose $Y_1(0)=-\beta<0$, where β is sufficiently large such that $Y_1(t)<0$ for all $t\le\pi/4$. As long as $Y_1(t)<0$ and $t\le\pi/4$ it holds that $\dot{Y}_1\le -3Y_1-Y_1^2(1/2)$. Hence, using the comparison principle we get that $Y_1(t)\le-(6e^{-3t}\beta/(6+\beta(e^{-3t}-1)))$. Choosing, for example, $\beta=12$ we have that $Y_1(t)\le-(12/(2-e^{3t}))$. Hence, $Y_1(t)<0$ for all $t\le\log(2)/3\approx 0.23<\pi/4\approx 0.79$. Moreover, $|Y_1|\to\infty$ before $t=\log(2)/3$ (which also implies that $|X|\to\infty$).

We employ next our predictor-feedback design when $0<D_1<D_2$. The predictor-feedback control law is given by

$$U_1(t)=-M_1(t)^2\cos(t+D_1)-M_1(t)Q_1(t)\times\left(1+\cos^2(t+D_1)\right)-P_{1_{X_3}}(t) \tag{89}$$

$$U_2(t)=-M_2(t)+Q_2(t)(\sin(t+D_2)-\cos(t+D_2))+Q_2(t)\left(-M_2(t)^2\cos(t+D_2)-M_2(t)Q_2(t)\times\left(1+\cos^2(t+D_2)\right)-P_{2_{X_3}}(t)\right) \tag{90}$$

where for $i = 1, 2$

$$M_i(t) = P_{i_{X_1}}(t)\cos\left(P_{i_{X_3}}(t)\right) + P_{i_{X_2}}(t)\sin\left(P_{i_{X_3}}(t)\right) \quad (91)$$

$$Q_i(t) = P_{i_{X_1}}(t)\sin\left(P_{i_{X_3}}(t)\right) - P_{i_{X_2}}(t)\cos\left(P_{i_{X_3}}(t)\right) \quad (92)$$

and $P_i = \begin{bmatrix} P_{i_{X_1}} \\ P_{i_{X_2}} \\ P_{i_{X_3}} \end{bmatrix}$, $i = 1, 2$. The predictors are given by

$$P_{1_{X_1}}(t) = X_1(t) + \int_{t-D_1}^{t} U_2(\theta - D_{21})\cos\left(P_{1_{X_3}}(\theta)\right) d\theta \quad (93)$$

$$P_{1_{X_2}}(t) = X_2(t) + \int_{t-D_1}^{t} U_2(\theta - D_{21})\sin\left(P_{1_{X_3}}(\theta)\right) d\theta \quad (94)$$

$$P_{1_{X_3}}(t) = X_3(t) + \int_{t-D_1}^{t} U_1(\theta)d\theta \quad (95)$$

$$P_{2_{X_1}}(t) = P_{1_{X_1}}(t) + \int_{t-D_{21}}^{t} U_2(\theta)\cos\left(P_{2_{X_3}}(\theta)\right) d\theta \quad (96)$$

$$P_{2_{X_2}}(t) = P_{1_{X_2}}(t) + \int_{t-D_{21}}^{t} U_2(\theta)\sin\left(P_{2_{X_3}}(\theta)\right) d\theta \quad (97)$$

$$P_{2_{X_3}}(t) = P_{1_{X_3}}(t) + \int_{t-D_{21}}^{t} \kappa_1\left(\theta + D_2, P_{2_{X_1}}(\theta), P_{2_{X_2}}(\theta), P_{2_{X_3}}(\theta)\right) d\theta \quad (98)$$

where for all $t - D_{21} \le \theta \le t$, $\kappa_1(\theta + D_2, P_{2_{X_1}}(\theta), P_{2_{X_2}}(\theta), P_{2_{X_3}}(\theta)) = -M_2(\theta)^2 \cos(\theta + D_2) - M_2(\theta)Q_2(\theta)(1 + \cos^2(\theta + D_2)) - P_{2_{X_3}}(\theta)$. Note that the D_1-time units ahead predictors, namely $P_{1_{X_i}}$, $i = 1, 2, 3$, given by (93)–(95) can be computed explicitly in terms of the history of $U_1(\theta)$ on the interval $\theta \in [t - D_1, t]$, the history of $U_2(\theta)$ on the interval $\theta \in [t - D_2, t - D_{2,1}]$, and the current states $X_i(t)$, $i = 1, 2, 3$, as

$$P_{1_{X_1}}(t) = X_1(t) + \int_{t-D_1}^{t} U_2(\theta - D_{2,1}) \times \cos\left(X_3(t) + \int_{t-D_1}^{\theta} U_1(s)ds\right) d\theta \quad (99)$$

$$P_{1_{X_2}}(t) = X_2(t) + \int_{t-D_1}^{t} U_2(\theta - D_{2,1}) \times \sin\left(X_3(t) + \int_{t-D_1}^{\theta} U_1(s)ds\right) d\theta \quad (100)$$

$$P_{1_{X_3}}(t) = X_3(t) + \int_{t-D_1}^{t} U_1(\theta)d\theta. \quad (101)$$

This is not possible for the D_2-time units ahead predictors, namely $P_{2_{X_i}}$, $i = 1, 2, 3$, given by (96)–(98), since in this case $P_{2_{X_3}}$ defined in (98) can not be solved explicitly in terms of the current states of the plant and the history of the actuator states.

VI. Application to Linear Systems

We specialize our control design to the case of linear systems in which case the control laws are given explicitly. We consider the following system:

$$\dot{X}(t) = AX(t) + \sum_{i=1}^{m} b_i U_i(t - D_i) \quad (102)$$

which can be written as

$$\dot{X}(t) = AX(t) + \sum_{i=1}^{m} b_i u_i(0, t) \quad (103)$$

$$\partial_t u_i(x, t) = \partial_x u_i(x, t), \quad x \in (0, D_i),\ i = 1, \ldots, m \quad (104)$$

$$u_i(D_i, t) = U_i(t), \quad i = 1, \ldots, m \quad (105)$$

where A is an $n \times n$ matrix and b_i, $i = 1, \ldots, m$ are n-dimensional vectors. System (103)–(105) is the linear version of system (13)–(15). We assume linear nominal feedback control laws, that is, in the delay-free case we have $U_i(t) = k_i^T X(t)$, and hence, Assumption 2 is satisfied with $\kappa_i(X) = k_i^T X$, $i = 1, \ldots, m$, under the assumption that the pair $(A, [b_1 \ \ldots \ b_m])$ is stabilizable. Note that Assumptions 1 and 3 hold for the case of linear systems under linear nominal feedback controllers. We first rewrite the predictor states in their PDE representation, namely (20)–(25), for the case of linear systems given by (103)–(105) as

$$\partial_x p_1(x, t) = Ap_1(x, t) + \sum_{i=1}^{m} b_i u_i(x, t), x \in [0, D_1] \quad (106)$$

$$\partial_x p_2(x, t) = \left(A + b_1 k_1^T\right) p_2(x, t) + \sum_{i=2}^{m} b_i u_i(x, t), \quad x \in [D_1, D_2] \quad (107)$$

$$\vdots$$

$$\partial_x p_m(x, t) = \left(A + \sum_{i=1}^{m-1} b_i k_i^T\right) p_m(x, t) + b_m u_m(x, t), \quad x \in [D_{m-1}, D_m] \quad (108)$$

with initial conditions given by (23)–(25). Solving explicitly the linear ODEs in x (106)–(108) and using the boundary conditions (23)–(25) we get that

$$p_1(x, t) = e^{Ax} X(t) + \int_0^x e^{A(x-y)} \sum_{i=1}^{m} b_i u_i(y, t)dy, \quad x \in [0, D_1] \quad (109)$$

$$p_2(x, t) = e^{A_1(x-D_1)} p_1(D_1, t) + \int_{D_1}^{x} e^{A_1(x-y)} \times \sum_{i=2}^{m} b_i u_i(y, t)dy, \quad x \in [D_1, D_2] \quad (110)$$

$$\vdots$$

$$p_m(x, t) = e^{A_{m-1}(x-D_{m-1})} p_{m-1}(D_{m-1}, t) + \int_{D_{m-1}}^{x} e^{A_{m-1}(x-y)} b_m u_m(y, t)dy \quad x \in [D_{m-1}, D_m] \quad (111)$$

where we used the notation

$$A_0 = A \tag{112}$$

$$A_i = A + \sum_{j=1}^{i} b_j k_j^T, \quad i = 1, \ldots, m. \tag{113}$$

The control laws are given by

$$U_i(t) = \kappa_i^T p_i(D_i, t), \quad i = 1, \ldots, m. \tag{114}$$

Theorem 3: Consider the closed-loop system consisting of the plant (103)–(105) and the control laws (114), (109)–(111). Let the pair $(A, [b_1 \ \ldots \ b_m])$ be stabilizable. There exist positive constants λ and μ such that for all initial conditions $X_0 \in \mathbb{R}^n$ and $u_{i_0} \in H^1(0, D_i)$, $i = 1, \ldots, m$, which are compatible with the feedback laws, that is, they satisfy $u_{i_0}(D_i) = \kappa_i^T p_i(D_i, 0)$, $i = 1, \ldots, m$, the closed-loop system has a unique solution $(X(t), u_1(\cdot, t), \ldots, u_m(\cdot, t)) \in C([0, \infty); \mathbb{R}^n \times H^1(0, D_1) \times \cdots \times H^1(0, D_m)) \cap C^1([0, \infty); \mathbb{R}^n \times L^2(0, D_1) \times \cdots \times L^2(0, D_m))$ and the following holds:

$$\Gamma(t) \leq \mu \Gamma(0) e^{-\lambda t}, \quad \text{for all } t \geq 0 \tag{115}$$

where

$$\Gamma(t) = |X(t)| + \sum_{i=1}^{m} \int_0^{D_i} u_i(x, t)^2 dx. \tag{116}$$

Note that with definitions $p_i(x, t) = P_i(t + x - D_i)$, for all $x \in [D_{i-1}, D_i]$, with $D_0 = 0$ (see Section III-B) the predictors (109)–(111) can be written as

$$P_1(\theta) = e^{A(\theta - t + D_1)} X(t) + \int_{t-D_1}^{\theta} e^{A(\theta - s)} \times \sum_{i=1}^{m} b_i U_i(s - D_{i,1}) ds, \quad t - D_1 \leq \theta \leq t \tag{117}$$

$$P_2(\theta) = e^{A_1(\theta - t + D_{2,1})} P_1(t) + \int_{t-D_{2,1}}^{\theta} e^{A_1(\theta - s)} \times \sum_{i=2}^{m} b_i U_i(s - D_{i,2}) ds, \quad t - D_{2,1} \leq \theta \leq t \tag{118}$$

$$\vdots$$

$$P_m(\theta) = e^{A_{m-1}(\theta - t + D_{m,m-1})} P_{m-1}(t) + \int_{t-D_{m,m-1}}^{\theta} e^{A_{m-1}(\theta - s)} \times b_m U_m(s) ds, \quad t - D_{m,m-1} \leq \theta \leq t \tag{119}$$

where the matrices A_i, $i = 1, \ldots, m-1$, are given in (112) and (113). With this notation the control laws (114) are written as

$$U_i(t) = \kappa_i^T P_i(t), \quad i = 1, \ldots, m. \tag{120}$$

We have the following corollary as an immediate consequence of Theorem and relation (16).

Corollary 2 (Theorem 3 in Standard Delay Notation): Consider the closed-loop system consisting of the plant (102) and the control laws (120), (117)–(119). Under the assumption that the pair $(A, [b_1 \ \ldots \ b_m])$ is stabilizable the following holds:

$$\Psi(t) \leq \mu \Psi(0) e^{-\lambda t}, \quad \text{for all } t \geq 0 \tag{121}$$

where

$$\Psi(t) = |X(t)| + \sum_{i=1}^{m} \int_{t-D_i}^{t} U_i(\theta)^2 d\theta. \tag{122}$$

Proof: We prove Theorem 3 by showing that the control laws (114), (109)–(111) are identical to the ones introduced in [55] and [56], and hence, Theorem 3 is proved by following the proof of Theorem 1 in [56]. Equivalently we show that the backstepping transformations (1) specialized to the linear case are identical to the backstepping transformations introduced in [56]. Toward that end define for all $i = 1, \ldots, m$

$$\phi_i(x) = \begin{cases} x, & x \leq D_i \\ D_i, & x \geq D_i \end{cases} \tag{123}$$

$$g_i(x, t) = \begin{cases} p_1(x, t), & x \in [0, D_1] \\ p_2(x, t), & x \in [D_1, D_2] \\ \vdots \\ p_i(x, t), & x \in [D_{i-1}, D_i]. \end{cases} \tag{124}$$

Then, using (109)–(111) it follows that:

$$e^{-A_{i-1}x} g_i(x, t) = v_{i-1}(x, t), \quad x \in [0, D_i] \quad i = 1, \ldots, m \tag{125}$$

where for all $x \in [0, D_m]$

$$v_i(x, t) = e^{-A_i \phi_i(x)} e^{A_{i-1} \phi_i(x)} v_{i-1}(x, t) + \int_{\phi_i(x)}^{\phi_{i+1}(x)} e^{-A_i y} \sum_{j=i+1}^{m} b_j u_j(y, t) dy \tag{126}$$

$$v_0(x, t) = X(t) + \int_0^{\phi_1(x)} e^{-Ay} \sum_{j=1}^{m} b_j u_j(y, t) dy. \tag{127}$$

We show this by induction. For all $x \in [0, D_1]$ it holds that

$$\begin{aligned} e^{-Ax} g_1(x, t) &= e^{-Ax} p_1(x, t) \\ &= X(t) + \int_0^{\phi_1(x)} e^{-Ay} \sum_{j=1}^{m} b_j u_j(y, t) dy \\ &= v_0(x) \end{aligned} \tag{128}$$

where we used (109) and (123). Assume now that (125) holds for some i. It follows that:

$$g_{i+1}(x, t) = \begin{cases} p_1(x, t), & x \in [0, D_1] \\ p_2(x, t), & x \in [D_1, D_2] \\ \vdots \\ p_i(x, t), & x \in [D_{i-1}, D_i] \\ p_{i+1}(x, t), & x \in [D_i, D_{i+1}] \end{cases} = \begin{cases} e^{A_{i-1}x} v_{i-1}(x, t), & x \in [0, D_i] \\ p_{i+1}(x, t), & x \in [D_i, D_{i+1}] \end{cases} \tag{129}$$

for all $x \in [0, D_{i+1}]$. Using (109)–(111) and (123) we get from (129) that

$$e^{-A_i x} g_{i+1}(x,t) = \begin{cases} e^{-A_i\phi_i(x)} e^{A_{i-1}\phi_i(x)} v_{i-1}(x,t), \\ x \in [0, D_i] \\ e^{-A_i D_i} p_i(D_i,t) + \int_{D_i}^{x} e^{-A_i y} \\ \times \sum_{j=i+1}^{m} b_j u_j(y,t)dy, \\ x \in [D_i, D_{i+1}]. \end{cases} \tag{130}$$

Using (124) and (125) it follows that:

$$\begin{aligned} e^{-A_{i-1}D_i} g_i(D_i,t) &= e^{-A_{i-1}D_i} p_i(D_i,t) \\ &= v_{i-1}(D_i,t) \end{aligned} \tag{131}$$

and hence

$$p_i(D_i,t) = e^{A_{i-1}D_i} v_{i-1}(D_i,t). \tag{132}$$

Combining (130) with (132) we arrive at

$$e^{-A_i x} g_{i+1}(x,t) = \begin{cases} e^{-A_i\phi_i(x)} e^{A_{i-1}\phi_i(x)} v_{i-1}(x,t) \\ x \in [0, D_i] \\ e^{-A_i D_i} e^{A_{i-1}D_i} v_{i-1}(D_i,t) \\ + \int_{D_i}^{x} e^{-A_i y} \sum_{j=i+1}^{m} b_j u_j(y,t)dy \\ x \in [D_i, D_{i+1}]. \end{cases} \tag{133}$$

We then observe from (123) that $\phi_i(x) = \phi_{i+1}(x) = x$, for all $x \le D_i$, and hence

$$\int_{\phi_i(x)}^{\phi_{i+1}(x)} e^{-A_i y} \sum_{j=i+1}^{m} b_j u_j(y,t)dy = 0, \quad x \in [0, D_i]. \tag{134}$$

Moreover, using (123) we get that

$$e^{-A_i D_i} e^{A_{i-1}D_i} = e^{-A_i\phi_i(x)} e^{A_{i-1}\phi_i(x)}, \; x \in [D_i, D_{i+1}]. \tag{135}$$

The proof is completed if we show that for all $i = 1, \ldots, m$

$$v_{i-1}(D_i,t) = v_{i-1}(x,t), \quad \text{for all } x \in [D_i, D_m]. \tag{136}$$

We prove this by induction and by using definitions (126), (127). Using (123) for $i = 1$ we have that

$$\begin{aligned} v_0(D_1,t) &= X(t) + \int_0^{D_1} e^{-Ay} \sum_{j=1}^{m} b_j u_j(y,t)dy \\ &= X(t) + \int_0^{\phi_1(x)} e^{-Ay} \sum_{j=1}^{m} b_j u_j(y,t)dy \quad x \ge D_1 \\ &= v_0(x,t), \quad \text{for all } x \ge D_1. \end{aligned} \tag{137}$$

Assume now that (136) holds for some i. Then using (123) and (126) we have that

$$\begin{aligned} v_i(D_{i+1},t) &= e^{-A_i D_i} e^{A_{i-1}D_i} v_{i-1}(D_{i+1},t) \\ &\quad + \int_{D_i}^{D_{i+1}} e^{-A_i y} \sum_{j=i+1}^{m} b_j u_j(y,t)dy \\ &= e^{-A_i\phi_i(x)} e^{A_{i-1}\phi_i(x)} v_{i-1}(D_{i+1},t) \\ &\quad + \int_{\phi_i(x)}^{\phi_{i+1}(x)} e^{-A_i y} \sum_{j=i+1}^{m} b_j u_j(y,t)dy \\ &\quad \text{for all } x \ge D_{i+1}. \end{aligned} \tag{138}$$

Therefore, using (136) we get that

$$\begin{aligned} v_i(D_{i+1},t) &= e^{-A_i\phi_i(x)} e^{A_{i-1}\phi_i(x)} v_{i-1}(x,t) \\ &\quad + \int_{\phi_i(x)}^{\phi_{i+1}(x)} e^{-A_i y} \sum_{j=i+1}^{m} b_j u_j(y,t)dy \\ &= v_i(x,t), \quad \text{for all } x \ge D_{i+1} \end{aligned} \tag{139}$$

which completes the proof that (125) holds. Therefore, using (124) the backstepping transformations (47)–(49) can be written in the linear case as

$$w_1(x,t) = u_1(x,t) - \kappa_1^T e^{A_0 x} v_0(x,t), \quad x \in [0, D_1] \tag{140}$$

$$w_2(x,t) = u_2(x,t) - \kappa_2^T e^{A_1 x} v_1(x,t), \quad x \in [0, D_2] \tag{141}$$

$$\vdots$$

$$w_m(x,t) = u_m(x,t) - \kappa_m^T e^{A_{m-1}x} v_{m-1}(x,t), \quad x \in [0, D_m] \tag{142}$$

which are identical to the backstepping transformations (67), (68) from [56].

As a special case we provide explicitly the first two backstepping transformations. Using (126) and (127), the backstepping transformations (140) and (141) take the form

$$w_1(x,t) = u_1(x,t) - k_1^T \left(e^{Ax} X(t) + \int_0^x e^{A(x-y)} \times \sum_{i=1}^{m} b_i u_i(y,t)dy \right), \quad x \in [0, D_1] \tag{143}$$

$$w_2(x,t) = u_2(x,t) - k_2^T \begin{cases} p_1(x,t), & x \in [0, D_1] \\ p_2(x,t), & x \in [D_1, D_2] \end{cases} \tag{144}$$

where

$$p_1(x,t) = e^{Ax}X(t) + \int_0^x e^{A(x-y)} \sum_{i=1}^m b_i u_i(y,t)dy \quad (145)$$

$$p_2(x,t) = e^{A_1(x-D_1)}e^{AD_1}X(t) + e^{A_1(x-D_1)} \times \int_0^{D_1} e^{A(D_1-y)} \sum_{i=1}^m b_i u_i(y,t)dy + \int_{D_1}^x e^{A_1(x-y)} \sum_{i=2}^m b_i u_i(y,t)dy \quad (146)$$

which are identical to relations (15), (77), and (78) from [56]. The predictor feedback control laws are

$$u_1(D_1,t) = k_1^T e^{AD_1}X(t) + k_1^T \int_0^{D_1} e^{A(D_1-y)} \times (b_1u_1(y,t) + b_2u_2(y,t))\, dy \quad (147)$$

$$u_2(D_2,t) = k_2^T \Big(e^{A_1D_{2,1}}e^{AD_1}X(t) + e^{A_1D_{2,1}} \int_0^{D_1} e^{A(D_1-y)} \times b_1u_1(y,t)dy + e^{A_1D_{2,1}} \int_0^{D_1} e^{A(D_1-y)}b_2u_2(y,t) \times dy + \int_{D_1}^{D_2} e^{A_1(D_2-y)}b_2u_2(y,t)dy \Big) \quad (148)$$

and in terms of the delayed states $U_i(t+x-D_i) = u_i(x,t)$ by

$$U_1(t) = k_1^T \Big(e^{AD_1}X(t) + \int_{t-D_1}^t e^{A(t-\theta)}b_1U_1(\theta)d\theta + \int_{t-D_2}^{t-D_{2,1}} e^{A(t-\theta-D_{2,1})}b_2U_2(\theta)d\theta \Big) \quad (149)$$

$$U_2(t) = k_2^T \Big(e^{A_1D_{2,1}}e^{AD_1}X(t) + e^{A_1D_{2,1}} \int_{t-D_1}^t e^{A(t-\theta)} \times b_1U_1(\theta)d\theta + e^{A_1D_{2,1}} \int_{t-D_2}^{t-D_{2,1}} e^{A(t-\theta-D_{2,1})}b_2 \times U_2(\theta)d\theta + \int_{t-D_{2,1}}^t e^{A_1(t-\theta)}b_2U_2(\theta)d\theta \Big). \quad (150)$$

Relations (149), (150) can be written as

$$U_1(t) = k_1^T P_1(t) \quad (151)$$

$$U_2(t) = k_2^T \Big(e^{A_1D_{2,1}}P_1(t) + \int_{t-D_{2,1}}^t e^{A_1(t-\theta)}b_2U_2(\theta)d\theta \Big) \quad (152)$$

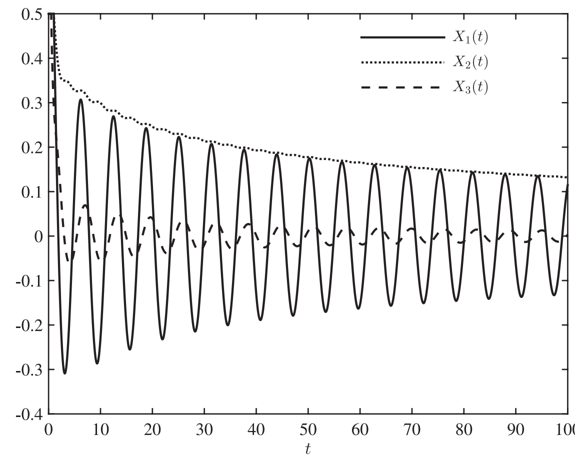

Fig. 3. The closed-loop response of system (82)–(84) with $D_1 = 0.5$ and $D_2 = 1$ under the predictor-feedback controller (89)–(98). The initial conditions of the plant are $X_1(0) = X_2(0) = X_3(0) = 0.5$ and for the actuator states are $U_1(\theta) = 0$, for all $-0.5 \le \theta \le 0$, and $U_2(\theta) = 0$, for all $-1 \le \theta \le 0$.

which are the control laws (46), (47) from [56]. Since the explicit expression of the general ith control law for the general case of m inputs is identical to the one obtained in [56] [relation (43)], for clarity of exposition we provide it again here

$$U_i(t) = k_i^T \Big(\Phi(D_i,0)X(t) + \sum_{j=1}^i \int_{t-D_j}^t \Phi(D_i, \theta - t + D_j) \times b_jU_j(\theta)d\theta + \sum_{j=i+1}^m \int_{t-D_j}^{t-D_{j,i}} \Phi(D_i, \theta - t + D_j) \times b_jU_j(\theta)d\theta \Big) \quad (153)$$

where for all $D_i \le x \le D_{i+1}$ and $D_j \le y \le \phi_{j+1}(x)$

$$\Phi(x,y) = e^{A_i(x-D_i)}e^{A_{i-1}(D_i-D_{i-1})} \ldots e^{A_j(D_{j+1}-y)} \quad (154)$$

for any $i, j \in \{0, 1, \ldots, m-1\}$ satisfying $0 \le i < j$, and

$$\Phi(x,y) = e^{A_i(x-y)}, \quad \text{for all } D_i \le x \le y \le D_{i+1} \quad (155)$$

for $i \in \{0, 1, \ldots, m-1\}$, where ϕ_i is defined in (123) and we adopt the notation $D_0 = 0$.

VII. Simulations

We consider the stabilization problem of the nonholonomic unicycle subject to distinct input delays from Section V whose dynamics are described by (82)–(84). We choose $D_1 = 0.5$ and $D_2 = 1$. The initial conditions for the plant are chosen as $X_1(0) = X_2(0) = X_3(0) = 0.5$ and for the actuator states as $U_1(\theta) = 0$, for all $-D_1 \le \theta \le 0$, and $U_2(\theta) = 0$, for all $-D_2 \le \theta \le 0$. In Fig. 3 we show the response of the closed-loop system under the predictor-feedback controller (89)–(98) and in Fig. 4 the corresponding control efforts. The predictor-feedback controller asymptotically stabilizes the nonholonomic unicycle. In particular, at $t = 1$ both delays are compensated and the system behaves as in the nominal, delay-free case. In contrast, at is is shown in Fig. 5, the closed-loop system under the nominal, uncompensated controller is unstable.

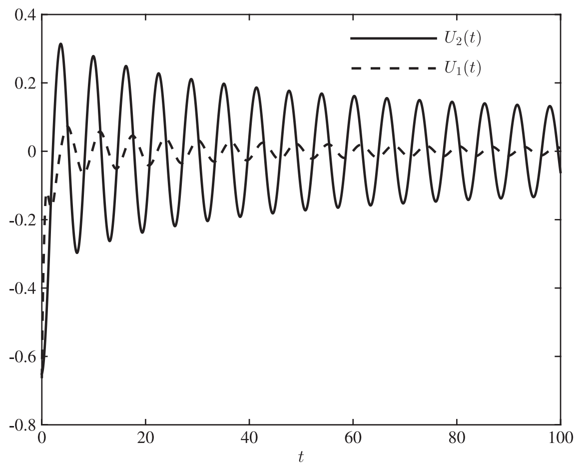

Fig. 4. The control efforts (89) and (90) of the closed-loop response of system (82)–(84) with $D_1 = 0.5$ and $D_2 = 1$ under the predictor-feedback controller (89)–(98). The initial conditions of the plant are $X_1(0) = X_2(0) = X_3(0) = 0.5$ and for the actuator states are $U_1(\theta) = 0$, for all $-0.5 \le \theta \le 0$, and $U_2(\theta) = 0$, for all $-1 \le \theta \le 0$.

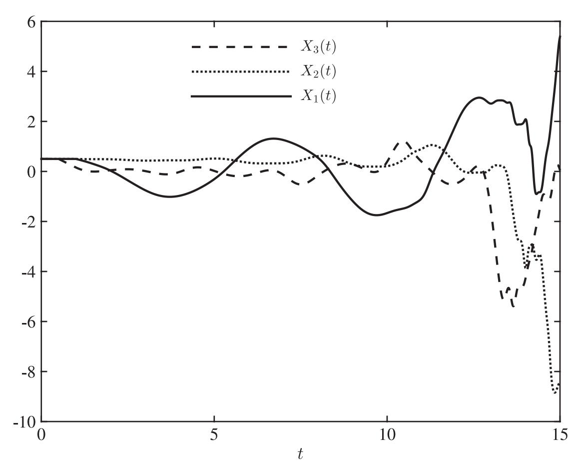

Fig. 5. The closed-loop response of system (82)–(84) with $D_1 = 0.5$ and $D_2 = 1$ under the nominal controller (85)–(88). The initial conditions of the plant are $X_1(0) = X_2(0) = X_3(0) = 0.5$ and for the actuator states are $U_1(\theta) = 0$, for all $-0.5 \le \theta \le 0$, and $U_2(\theta) = 0$, for all $-1 \le \theta \le 0$.

VIII. Conclusion

We presented a predictor-feedback control design methodology for the stabilization of multi-input nonlinear systems with distinct input delays. We proved global asymptotic stability of the closed-loop system using Lyapunov arguments and arguments based on estimates of closed-loop solutions. We also dealt with linear systems as a special case of our methodology. We applied our approach to the stabilization of the nonholonomic unicycle with delayed inputs.

In contrast to the single-input case, for which the predictor-feedback controller is available explicitly for some classes of nonlinear systems (see, for instance, [27]) with a specific open-loop structure, in the multi-input case the class of nonlinear systems for which the predictor-feedback control law can be obtained explicitly seems to be more restrictive (although it can be obtained explicitly in some trivial cases, such as, for example, the case of linear systems). This is attributed to the fact that the formulae of the predictors in the multi-input case depends not only on the open-loop structure of the system but also on the form of the feedback functions.

As a next step, it is of interest to study the problem of stabilization of multi-input nonlinear systems with actuator dynamics governed by wave or diffusion PDEs with different wave propagation speeds or diffusion coefficients, respectively, in each individual input channel. The starting point for such a study is [12].

Appendix A

Proof of Lemma 1: Setting $x = 0$ into (47)–(49), and (17) we get (50). Using (26) one can conclude that $\partial_t p_1(x,t) = \partial_x p_1(x,t)$, for all $x \in [0, D_1]$. Hence, using (14) and (47) we get that $\partial_t w_1(x,t) - \partial_x w_1(x,t) = \partial_t u_1(x,t) - \partial_x u_1(x,t) + (\partial\kappa_1(p_1(x,t))/\partial p)(\partial_t p_1(x,t) - \partial_x p_1(x,t)) = 0$. Similarly, using the fact that $p_i(x,t) = X(t+x)$, for all $x \in [D_{i-1}, D_i]$ and $i = 2, \ldots, m$, we get (51). Setting $x = D_1$ into (47)–(49) and using (15), (42) we arrive at (52).

Proof of Lemma 2: We prove this lemma by showing that $p_1(x,t) = \pi_1(x,t)$, for all $x \in [0, D_1]$, and $p_i(x,t) = \pi_i(x,t)$, for all $x \in [D_{i-1}, D_i]$, $i = 2, \ldots, m$. Equivalently we show that $\pi_i(x,t) = X(t+x)$, $i = 1, \ldots, m$ and use the fact that $p_i(x,t) = X(t+x)$, $i = 1, \ldots, m$. We first observe that $\pi_1(0,t) = X(t)$, and hence, π_1 satisfies the following initial value problem for all $x \in [0, D_1]$:

$$\partial_x \pi_1(x,t) = f\left(\pi_1(x,t), w_1(x,t) + \kappa_1(\pi_1(x,t)), \ldots, w_m(x,t) + \kappa_m\left(\pi_1(x,t)\right)\right) \tag{A.1}$$

$$\pi_1(0,t) = X(t). \tag{A.2}$$

The solution to (A.1) and (A.2) is $\pi_1(x,t) = X(t+x)$. This solution satisfies the boundary condition (A.2). It also satisfies the ODE (A.1) since by (50) and (51) it follows that the following holds for all $t \ge 0$ and $0 \le x \le D_1$ $X'(t+x) = f(X(t+x), W_1(t+x-D_1) + \kappa_1(X(t+x)), \ \ldots, W_m(t+x-D_m) + \kappa_m(X(t+x)))$, where the solutions to (51) and (52) are given by $w_i(x,t) = W(t+x-D_i)$, for all $x \in [0, D_i]$ and $i = 1, \ldots, m$, where W_i, $i = 1, \ldots, m$, satisfy $W_i(t) = 0$, for all $t \ge 0$. Assume now that $\pi_j(x,t) = X(t+x)$, for all $x \in [D_{j-1}, D_j]$, and some j. Then we claim that $\pi_{j+1} = X(t+x)$, for all $x \in [D_j, D_{j+1}]$. The function π_{j+1} satisfies the following initial value problem:

$$\begin{aligned}\partial_x \pi_{j+1}(x,t) = f\big(&\pi_{j+1}(x,t), \kappa_1\left(\pi_{j+1}(x,t)\right), \ldots \\ &\kappa_j\left(\pi_{j+1}(x,t)\right), w_{j+1}(x,t) \\ &+ \kappa_{j+1}\left(\pi_{j+1}(x,t)\right), \ldots, w_m(x,t) \\ &+ \kappa_m(\pi_{j+1}(x,t))\big), \quad x \in [D_j, D_{j+1}]\end{aligned} \tag{A.3}$$

$$\pi_{j+1}(D_j,t) = \pi_j(D_j,t). \tag{A.4}$$

Since $\pi_j(x,t) = X(t+x)$, for all $x \in [D_{j-1}, D_j]$, it follows that $\pi_{j+1}(x,t) = X(t+x)$, $x \in [D_j, D_{j+1}]$, satisfies the boundary condition (A.4). Using (51) and (52) we get that

$w_i(x,t) = W_i(t+x-D_i)$, for all $x \in [0, D_i]$, where $W_i(t) = 0$, for all $t \geq 0$. Since from (50) it holds that

$$X'(t+x) = f\left(X(t+x), W_1(t+x-D_1) + \kappa_1\left(X(t+x)\right), \ldots, W_m(t+x-D_m) + \kappa_m\left(X(t+x)\right)\right) \quad \text{(A.5)}$$

for all $t \geq 0$ and $x \geq 0$, one can conclude that the following hods for all $t \geq 0$ and $x \in [D_j, D_{j+1}]$:

$$X'(t+x) = f\left(X(t+x), \kappa_1\left(X(t+x)\right), \ldots, \kappa_j\left(X(t+x)\right) W_{j+1}(t+x-D_{j+1}) + \kappa_{j+1}\left(X(t+x)\right), \ldots W_m(t+x-D_m) + \kappa_m\left(X(t+x)\right)\right) \quad \text{(A.6)}$$

and hence, $\pi_{j+1} = X(t+x)$, for all $x \in [D_j, D_{j+1}]$. Since this holds for an arbitrary j one can conclude that $\pi_j = X(t+x)$, for all $x \in [D_{j-1}, D_j]$ and $j = 1, \ldots, m$, with $D_0 = 0$, which completes the proof.

Proof of Lemma 3: From Assumption 2 it follows from [54] that there exist a smooth function $S(Z): \mathbb{R}^{n+1} \to \mathbb{R}_+$ and class $\mathcal{K}_\infty$ functions α_1, α_2, α_3, and α_4, such that:

$$\alpha_1\left(|X|\right) \leq S(X) \leq \alpha_2\left(|X|\right) \quad \text{(A.7)}$$

$$F(X, \omega_1, \ldots, \omega_m) \leq -\alpha_3\left(|X|\right) + \alpha_4\left(\sum_{i=1}^m |\omega_i|\right) \quad \text{(A.8)}$$

$$F(X, \omega_1, \ldots, \omega_m) = \frac{\partial S(X)}{\partial X} f\left(X, \kappa_1(X) + \omega_1, \ldots, \kappa_m(X) + \omega_m\right) \quad \text{(A.9)}$$

for all $X \in \mathbb{R}^n$ and $\omega_1, \ldots, \omega_m \in \mathbb{R}$. With similar calculations as in [37] (Theorem 5) we get that

$$\frac{d\left\|w_i(\cdot,t)\right\|_{c,\infty}}{dt} \leq -c\left\|w_i(\cdot,t)\right\|_{c,\infty}, \quad i = 1, \ldots, m \quad \text{(A.10)}$$

along the solutions of (51) and (52). Consider the Lyapunov functional

$$V(t) = S\left(X(t)\right) + \frac{2}{c}\int_0^{\sum_{i=1}^m \|w_i(\cdot,t)\|_{c,\infty}} \frac{\alpha_4(r)}{r} dr. \quad \text{(A.11)}$$

Using Lemma 1 [relation (50)] and relations (A.8), (A.10) we get along the solutions of (50)–(52) that

$$\dot{V}(t) \leq -\alpha_3\left(|X(t)|\right) + \alpha_4\left(\sum_{i=1}^m |w_i(0,t)|\right) - 2\alpha_4\left(\sum_{i=1}^m \left\|w_i(\cdot,t)\right\|_{c,\infty}\right). \quad \text{(A.12)}$$

Using the fact that $|w_i(0,t)| \leq \|w_i(\cdot,t)\|_{c,\infty}$, $i = 1, \ldots, m$, we get that

$$\dot{V}(t) \leq -\alpha_3\left(|X(t)|\right) - \alpha_4\left(\sum_{i=1}^m \left\|w_i(\cdot,t)\right\|_{c,\infty}\right). \quad \text{(A.13)}$$

It follows with the help of (A.7) that there exists a class $\mathcal{K}$ function γ_1 such that:

$$\dot{V}(t) \leq -\gamma_1\left(V(t)\right) \quad \text{(A.14)}$$

and hence, with the comparison principle (see, for example, [33]) one can conclude that there exists a class $\mathcal{KL}$ function β_2 such that

$$V(t) \leq \beta_2\left(V(0), t\right). \quad \text{(A.15)}$$

Using (A.7), the fact that $\|w_i(\cdot,t)\|_\infty \leq \|w_i(\cdot,t)\|_{c,\infty} \leq e^{cD_i}\|w_i(\cdot,t)\|_\infty$, $i = 1, \ldots, m$, and the properties of class $\mathcal{K}$ functions we get estimate (59).

Proof of Lemma 4: We prove the lemma by induction. For clarity we present two initial steps. We prove first bound (61). Using the fact that p_1 satisfies (20) we get under Assumption 1 that there exists a smooth function $R: \mathbb{R}^n \to \mathbb{R}_+$ and class $\mathcal{K}_\infty$ functions α_5, α_6, and α_7 such that

$$\alpha_5\left(|X|\right) \leq R(X) \leq \alpha_6\left(|X|\right) \quad \text{(A.16)}$$

$$\frac{\partial R(X)}{\partial X} f(X, \omega_1, \ldots, \omega_m) \leq R(X) + \alpha_7\left(\sum_{i=1}^m |\omega_i|\right) \quad \text{(A.17)}$$

for all $X \in \mathbb{R}^n$ and $\omega_i \in \mathbb{R}$, $i = 1, \ldots, m$. Therefore

$$\frac{dR(p_1(x,t))}{dx} = \frac{\partial R(p_1(x,t))}{\partial p_1} f\left(p_1(x,t), u_1(x,t), \ldots, u_m(x,t)\right) \leq R\left(p_1(x,t)\right) + \alpha_7\left(\sum_{i=1}^m |u_i(x,t)|\right) \quad x \in [0, D_1]. \quad \text{(A.18)}$$

Hence, using (23) we get that

$$R\left(p_1(x,t)\right) \leq e^{D_1} R\left(X(t)\right) + e^{D_1}\int_0^{D_1} \alpha_7\left(\sum_{i=1}^m |u_i(x,t)|\right) dx \quad \text{(A.19)}$$

and hence

$$R\left(p_1(x,t)\right) \leq e^{D_1} R\left(X(t)\right) + D_1 e^{D_1}\alpha_7\left(\sum_{i=1}^m \left\|u_i(\cdot,t)\right\|_\infty\right). \quad \text{(A.20)}$$

With the help of (A.16) and the properties of class $\mathcal{K}$ functions we get estimate (61). We prove next (62). Under Assumption 3 one can conclude that there exists a smooth function $R_1: \mathbb{R}^n \to \mathbb{R}_+$ and class $\mathcal{K}_\infty$ functions α_8, α_9, and α_{10} such that

$$\alpha_8\left(|X|\right) \leq R_1(X) \leq \alpha_9\left(|X|\right) \quad \text{(A.21)}$$

$$\frac{\partial R_1(X)}{\partial X} g_1(X, \omega_2, \ldots, \omega_m) \leq R_1(X) + \alpha_{10}\left(\sum_{i=2}^m |\omega_i|\right) \quad \text{(A.22)}$$

for all $X \in \mathbb{R}^n$ and $\omega_i \in \mathbb{R}$, $i = 2, \ldots, m$, where g_1 is defined in Assumption 3. Using the fact that p_2 satisfies (21) we get from (A.22) that

$$\begin{aligned}\frac{dR_1\left(p_2(x,t)\right)}{dx} =& \frac{\partial R_1\left(p_2(x,t)\right)}{\partial p} g_1\left(p_2(x,t), u_2(x,t)\right.\\ & \left. \ldots, u_m(x,t)\right)\\ \leq& R_1\left(p_2(x,t)\right) + \alpha_{10}\left(\sum_{i=2}^{m} |u_i(x,t)|\right)\\ & x \in [D_1, D_2]. \end{aligned} \tag{A.23}$$

Thus, using (24) we get that

$$R_1\left(p_2(x,t)\right) \leq e^{D_2} R_1\left(p_1(D_1,t)\right) + e^{D_2} \times \int_0^{D_2} \alpha_{10}\left(\sum_{i=2}^{m} |u_i(x,t)|\right) dx \tag{A.24}$$

and hence, from (A.21) that

$$R_1\left(p_2(x,t)\right) \leq e^{D_2} \alpha_9\left(\|p_1(t)\|_\infty\right) + D_2 e^{D_2} \times \alpha_{10}\left(\sum_{i=2}^{m} \|u_i(\cdot,t)\|_\infty\right). \tag{A.25}$$

Using (A.21) and (61) we get from (A.25) estimate (62). Assume now that for some j it holds that

$$\|p_j(\cdot,t)\|_\infty \leq \rho_j\left(\Xi(t)\right). \tag{A.26}$$

Under Assumption 3 there exist a smooth function $R_j : \mathbb{R}^n \to \mathbb{R}_+$ and class $\mathcal{K}_\infty$ functions $\alpha_{8+3(j-1)}$, $\alpha_{9+3(j-1)}$, and $\alpha_{10+3(j-1)}$ such that

$$\alpha_{8+3(j-1)}\left(|X|\right) \leq R_j(X) \leq \alpha_{9+3(j-1)}\left(|X|\right) \tag{A.27}$$

$$G_j(X, \omega_{j+1}, \ldots, \omega_m) \leq R_j(X) + \alpha_{10+3(j-1)}\left(\sum_{i=j+1}^{m} |\omega_i|\right) \tag{A.28}$$

$$G_j(X, \omega_{j+1}, \ldots, \omega_m) = \frac{\partial R_j(X)}{\partial X} g_j(X, \omega_{j+1}, \ldots, \omega_m) \tag{A.29}$$

for all $X \in \mathbb{R}^n$ and $\omega_i \in \mathbb{R}$, $i = j+1, \ldots, m$. Using the fact that p_{j+1} satisfies (37) and definition $g_j(X, \omega_{j+1}, \ldots, \omega_m) = f(X, \kappa_1(X), \ldots, \kappa_j(X), \omega_{j+1}, \ldots, \omega_m)$ we get from (A.28) that

$$\begin{aligned}\frac{dR_j\left(p_{j+1}(x,t)\right)}{dx} =& \frac{\partial R_j\left(p_{j+1}(x,t)\right)}{\partial p} g_j\left(p_{j+1}(x,t)\right.\\ & \left. u_{j+1}(x,t), \ldots, u_m(x,t)\right)\\ \leq& R_j\left(p_{j+1}(x,t)\right)\\ & + \alpha_{10+3(j-1)}\left(\sum_{i=j+1}^{m} |u_i(x,t)|\right)\\ & x \in [D_j, D_{j+1}]. \end{aligned} \tag{A.30}$$

Therefore, employing (A.27) we get that

$$R_j\left(p_{j+1}(x,t)\right) \leq e^{D_{j+1}} \alpha_{9+3(j-1)}\left(\|p_j(t)\|_\infty\right) + D_{j+1} e^{D_{j+1}} \times \alpha_{10+3(j-1)}\left(\sum_{i=j+1}^{m} \|u_i(\cdot,t)\|_\infty\right). \tag{A.31}$$

Using (A.27), (A.26), and the properties of class $\mathcal{K}$ functions the proof is completed.

Proof of Lemma 5: The proof of this lemma employs similar arguments to the proof of Lemma 4 with the difference that one uses the ODEs in x for π_i, $i = 1, \ldots, m$ together with Assumption 2. We again prove this lemma by induction. We first prove (64). Using the fact that π_1 satisfies the initial value problem (A.1), (A.2), we get under Assumption 2 and the definition of input-to-state stability (see, for example, [54]) that there exist a class $\mathcal{KL}$ function β_3 and a class $\mathcal{K}_\infty$ function γ_2 such that

$$|\pi_1(x,t)| \leq \beta_3\left(|X(t)|, x\right) + \gamma_2\left(\sup_{0 \leq y \leq x}\left(\sum_{i=1}^{m} |w_i(x,t)|\right)\right) \quad x \in [0, D_1] \tag{A.32}$$

and hence, we arrive at (64) with $\bar{\rho}_1(s) = \beta_3(s,0) + \gamma_2(s)$. Assume next that for some j it holds that

$$\|\pi_j(\cdot,t)\|_\infty \leq \bar{\rho}_j\left(\bar{\Xi}(t)\right). \tag{A.33}$$

Using the fact that π_{j+1} satisfies the initial value problem (A.3), (A.4) we get under Assumption 2 that

$$\begin{aligned}|\pi_{j+1}(x,t)| \leq& \beta_3\left(|\pi_j(D_j)|, x - D_j\right)\\ & + \gamma_2\left(\sup_{D_j \leq y \leq x}\left(\sum_{i=j+1}^{m} |w_i(x,t)|\right)\right)\\ & x \in [D_j, D_{j+1}] \end{aligned} \tag{A.34}$$

and hence, with (A.33) we arrive at $\|\pi_{j+1}(\cdot,t)\|_\infty \leq \bar{\rho}_{j+1}(\bar{\Xi}(t))$, with $\bar{\rho}_{j+1}(s) = \beta_3(\bar{\rho}_j, 0) + \gamma_2(s)$.

Proof of Lemma 6: We prove first (67). Using the fact that κ_i, $i = 1, \ldots, m$, are locally Lipschitz with $\kappa_i(0) = 0$,

$i = 1, \ldots, m$, there exist class $\mathcal{K}_\infty$ functions α_i^*, $i = 1, \ldots, m$, such that

$$|\kappa_i(X)| \leq \alpha_i^*\left(|X|\right), \quad i = 1, \ldots, m \tag{A.35}$$

for all $X \in \mathbb{R}^n$. Hence, using (47)–(49) and relations (61)–(63) from Lemma 4 we get estimate (67) with $\rho(s) = s + \sum_{i=1}^m \alpha_i^*(\sum_{j=1}^m \rho_j(s))$. Similarly, using (53)–(55) we get estimate (68) by employing Lemma 5 with $\bar{\rho}(s) = s + \sum_{i=1}^m \alpha_i^*(\sum_{j=1}^m \bar{\rho}_j(s))$.

Appendix B
Verification of Assumptions 1 and 3 for the System Considered in Section V

The fact that system $\dot{X} = f(X, \omega_1, \omega_2)$, where

$$f(X, \omega_1, \omega_2) = \begin{bmatrix} \omega_2 \cos(X_3) \\ \omega_2 \sin(X_3) \\ \omega_1 \end{bmatrix} \tag{B.1}$$

satisfies Assumption 1 follows immediately by defining:

$$R(X) = \frac{1}{2} \sum_{i=1}^{3} X_i^2 \tag{B.2}$$

and readily verifying by employing Young's inequality that

$$\begin{aligned} \frac{\partial R(X)}{\partial X} f(X, \omega_1, \omega_2) &= X_1 \omega_2 \cos(X_3) + X_2 \omega_2 \sin(X_3) \\ &\quad + X_3 \omega_1 \\ &\leq R(X) + \frac{1}{2}\left(\omega_1^2 + \omega_2^2\right). \end{aligned} \tag{B.3}$$

Moreover, under (85) the dynamics of the nominal, delay-free system are described by $\dot{X} = f(X, \kappa_1(t, X), \omega_2)$, where

$$f\left(X, \kappa_1(t, X), \omega_2\right) = \begin{bmatrix} \omega_2 \cos(X_3) \\ \omega_2 \sin(X_3) \\ g(X, t) \end{bmatrix} \tag{B.4}$$

$$\begin{aligned} g(X, t) = &-X_3 - M(X)^2 \cos(t) \\ &- M(X) Q(X)\left(1 + \cos^2(t)\right). \end{aligned} \tag{B.5}$$

Therefore, by defining $S_2(X_3) = (1/2) X_3^2$ we have that

$$\frac{\partial S_2(X_3)}{\partial X_3} g(X, t) \leq 4\left(X_1^4 + X_2^4\right) \tag{B.6}$$

where we employed Young's inequality and the fact that $M^2 + Q^2 = X_1^2 + X_2^2$ which follows from (87) and (88). With $S_1(X_1, X_2) = (1/4)(X_1^4 + X_2^4)$ we get that $(\partial S_1(X_1, X_2)/\partial X_1)\omega_2 \cos(X_3) + (\partial S_1(X_1, X_2)/\partial X_2)\, \omega_2 \sin(X_3) \leq 3S_1 + (1/2)\omega_2^4$. Hence, defining $S = S_1 + S_2$ for system $\dot{X} = f(X,]\kappa_1(t, X), \omega_2)$ with (B.4) it follows that:

$$\begin{aligned} \frac{\partial S(X)}{\partial X} f(X, \kappa_1(t, X), \omega_2) &\leq 19 S_1 + \frac{1}{2}\omega_2^4 \\ &\leq 19 S + \frac{1}{2}\omega_2^4 \end{aligned} \tag{B.7}$$

and hence, it follows that system $\dot{X} = f(X, \kappa_1(t, X), \omega_2)$ with (B.4) satisfies Assumption 3.

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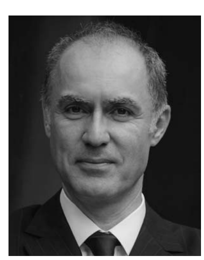
Nikolaos Bekiaris-Liberis received the Ph.D. degree from the University of California, San Diego, CA, USA, in 2013.

From 2013 to 2014, he was a Postdoctoral Researcher at the University of California, Berkeley, CA, USA. He is currently a Postdoctoral Researcher at the Dynamic Systems and Simulation Laboratory, Technical University of Crete, Crete, Greece. He has coauthored the SIAM book *Nonlinear Control under Nonconstant Delays*. His research interests include delay systems, distributed parameter systems, nonlinear control, and their applications.

Dr. Bekiaris-Liberis was a finalist for the Student Best Paper Award at the 2010 ASME Dynamic Systems and Control Conference and at the 2013 IEEE Conference on Decision and Control. He received the Chancellor's Dissertation Medal in Engineering from the University of California, San Diego, in 2014. He received the Best Paper Award in the 2015 International Conference on Mobile Ubiquitous Computing, Systems, Services, and Technologies.



Miroslav Krstic holds the Alspach Endowed Chair and is the Founding Director of the Cymer Center for Control Systems and Dynamics at UC San Diego, San Diego, CA, USA. He also serves as Associate Vice Chancellor for Research at UCSD. He has held the Springer Visiting Professorship at UC Berkeley. He has coauthored ten books on adaptive, nonlinear, and stochastic control, extremum seeking, control of PDE systems including turbulent flows, and control of delay systems.

As a graduate student, he won the UC Santa Barbara Best Dissertation Award and Student Best Paper Awards at CDC and ACC. He is a Fellow of IFAC, ASME, and IET (UK), and a Distinguished Visiting Fellow of the Royal Academy of Engineering. He has received the PECASE, NSF Career, and ONR Young Investigator awards, the Axelby and Schuck paper prizes, the Chestnut textbook prize, and the first UCSD Research Award given to an engineer. He serves as a Senior Editor for the IEEE TRANSACTIONS ON AUTOMATIC CONTROL and *Automatica*, as Editor of two Springer book series, and has served as Vice President for Technical Activities of the IEEE Control Systems Society and as Chair of the IEEE CSS Fellow Committee.