Control of Transport PDE/Nonlinear ODE Cascades With State-Dependent Propagation Speed

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***Abstract*—In this paper, we deal with the control of a transport partial differential equation/ nonlinear ordinary differential equation (PDE/nonlinear ODE) cascade system in which the transport coefficient depends on the ODE state. We develop a PDE-based predictor-feedback boundary control law, which compensates the transport dynamics of the actuator and guarantees global asymptotic stability of the closed-loop system. The stability proof is based on an infinite-dimensional backstepping transformation and a Lyapunov-like argument. The relation of the PDE–ODE cascade with a state-dependent propagation speed to an ODE system with a state-dependent input delay, which is defined implicitly via an integral of past values of the ODE state, is also highlighted and the corresponding equivalent predictor-feedback design is presented together with an alternative proof of global asymptotic stability of the closed-loop system based on the construction of a Lyapunov functional. The practical relevance of our control framework is illustrated in an example that is concerned with the control of a metal rolling process.**

***Index Terms*—Boundary control, metal rolling, nonlinear control, PDE-ODE cascade systems, predictor-feedback, state-dependent delay.**

I. Introduction

The problem of stabilization of coupled transport partial differential equation/ordinary differential equation (PDE/ODE) systems in which the transport coefficient or the boundary of the PDE domain varies with time is currently attracting considerable attention. This is attributed to the fact that such systems occur in a large number of challenging engineering problems, typically when sensors and actuators are not co-located and, particularly, in systems involving transport of materials. Among several other applications, such systems are utilized to describe the dynamics of screw extrusion processes for additive manufacturing [1], metal cutting processes [2], moisture in convective flows [3], populations [4], transport phenomena in gasoline engines [5]–[9], crushing-mills [10], production of commercial fuels by blending [11], and of stick-slip instabilities during oil drilling [12]–[15].

In this paper, we consider a particular class of implicitly defined state-dependent delays, which appear in numerous applications and which are expressed as transport, with a variable velocity (that may depend on the state of the ODE system), over a constant distance [16]. In engineering, such delays are sometimes called variable transport delays [17] and can be found, for example, in material flows in reactors [18], whereas the same type of delays can be even found in biology, typically known as threshold delays [4], [19].

Predictor-feedback control laws are often employed for compensation of constant input delays, which appear in numerous linear [20], [21] and nonlinear [22], [23] physical systems. In recent years, the extension of the predictor-feedback concept to the case of nonlinear systems with input delays that vary with time has been developed in [24]–[26] (see also [27], [28] for other prediction-based approaches for linear systems). Such predictor feedbacks are employed for stabilization of PDE–ODE cascades in which the PDE part describes the actuator dynamics of the ODE system, exploiting an alternative representation of the PDE–ODE system via a nonlinear system with input delay. In particular, the control of the nozzle flow rate of screw extruders in additive manufacturing is dealt in [1] utilizing a transport PDE/ODE cascade model in which the length of the PDE domain depends on the ODE state, and the stabilization of nonlinear systems is dealt in [12], [13] with actuator dynamics governed by a wave PDE with moving boundary that depends on the ODE state as well. Predictor-based control designs are also developed for stabilization of transport PDE–ODE cascades with input-dependent transport coefficient [9].

In this paper, we develop a PDE-based predictor-feedback law for stabilization of a transport PDE/nonlinear ODE cascade with state-dependent propagation speed. We prove global asymptotic stability of the closed-loop system with the aid of a Lyapunov functional that is constructed by introducing a novel infinite-dimensional backstepping transformation. An alternative repre-

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sentation of the PDE/ODE cascade as a nonlinear system with state-dependent input delay defined implicitly through an integral of the ODE state, is derived, computing the PDE solution with the method of characteristics. The equivalent predictor-feedback design for the delay system is also presented. We prove global asymptotic stability of the closed-loop system in the new representation providing an alternative proof.

The problem in this paper differs than a problem with a past-state-dependent delay [29], in that the different (in comparison to [29]) definition of the delay in the current case gives rise to a different prediction horizon, which is defined implicitly through an integral equation that incorporates the future values of the state over the entire prediction window (and not just as an explicit function of the current state as in the case of a past-state-dependent delay). This results in a different definition of the predictor in comparison to [29]. In addition, unlike the contributions [25], [29], this work offers a global stability result. This is due to the fact that the feasibility condition that the delay rate is less than one is satisfied *a priori* (irrespective of the values of the state and input). This is guaranteed by the assumption of the uniform (with respect to the state) strict positiveness of the transport speed made here, which imposes a single direction of propagation of the control signal along the actuation path (i.e., the control signal never propagates in the opposite direction).

The effectiveness of the proposed control approach is illustrated in a simulation of a model for the control of a metal rolling process [30]–[32], where a state-dependent delay due to a state-dependent transport velocity occurs [33], [34].

This paper is organized as follows: In Section II, the PDE/nonlinear ODE cascade system and the controller design are presented. The statement of the main result and the stability proof via PDE representation are introduced in Section III. Section IV discusses the alternative representation of the PDE/nonlinear ODE cascade system as an implicit state-dependent input delay system. The design of an equivalent controller for the delay system is established in Section V. The stability analysis via delay system representation is presented in Section VI. The paper ends with simulation results, which illustrate the practical relevance of the proposed framework via an application to metal rolling processes in Section VII. Final remarks and future directions are provided in Section VIII.

Notation: We use the common definition of class $\mathcal{K}$, $\mathcal{K}_\infty$, and $\mathcal{KL}_\infty$ from [35]. For an n-vector, the norm $|.|$ denotes the usual Euclidean norm. We denote by $C^j(A)$ the space of functions that have continuous derivatives of order j on A.

II. Problem Statement and Controller Design

We consider the transport PDE/nonlinear ODE cascade system with state-dependent propagation speed defined as

$$\dot{X}(t) = f\left(X(t), u(0,t)\right) \tag{1}$$

where $X \in \mathbb{R}^n$ and $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is continuously differentiable with $f(0,0) = 0$. The plant is located at the boundary $x = 0$ of a transport device (e.g., a pipe, which represents the actuation path) and controlled through a transport equation given

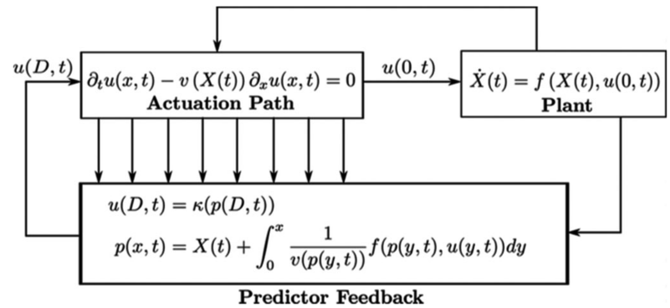

Fig. 1. Schematic of the closed-loop system.

by

$$\partial_t u(x,t) - v\left(X(t)\right)\partial_x u(x,t) = 0 \tag{2}$$

where $v : \mathbb{R}^n \to \mathbb{R}_+$ is continuously differentiable with respect to X. The actuation $U(t)$ at the boundary $x = D$ of the PDE is written as

$$u(D,t) = U(t). \tag{3}$$

The initial condition along the actuation path is defined as

$$u(x,0) = u_0(x). \tag{4}$$

Assumption 1: The state-dependent propagation speed $v : \mathbb{R}^n \to \mathbb{R}_+$ is continuously differentiable and there exists a positive constant $v_\star$, such that

$$v(X) \geq v_\star, \quad \text{for all } \ X \in \mathbb{R}^n. \tag{5}$$

Assumption 2: There exist a smooth positive definite function C and class $\mathcal{K}_\infty$ functions μ_1, μ_2, and μ_3, such that for the plant $\dot{X} = f(X, w)$, the following holds:

$$\mu_1(|X|) \leq C(X) \leq \mu_2(|X|) \tag{6}$$

$$\frac{\partial C(X)}{\partial X} f(X,\omega) \leq C(X) + \mu_3(|\omega|) \tag{7}$$

for all $(X,\ \omega)^T \in \mathbb{R}^{n+1}$.

Assumption 2 guarantees that system $\dot{X} = f(X,\omega)$ is strongly forward complete with respect to ω.

Assumption 3: System $\dot{X} = f(X, \kappa(X) + \omega)$ is input-to-state stable with respect to ω. Moreover, the feedback law $\kappa : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable with $\kappa(0) = 0$.

The definitions of strong forward completeness and input-to-state stability are those from [36] and [37], respectively.

The predictor-feedback controller for systems (1)–(3) is given by

$$u(D,t) = \kappa(p(D,t)) \tag{8}$$

$$p(x,t) = X(t) + \int_0^x \frac{1}{v(p(y,t))} f(p(y,t), u(y,t))dy \tag{9}$$

for all $x \in [0, D]$. We emphasize that for implementing control law (8), (9), one needs to measure the ODE state $X(t)$ and the PDE state $u(x,t)$, $x \in [0, D]$.

III. Main Result and its Proof via PDE Representation

Theorem 1: Consider the closed-loop system consisting of the plant (1)–(3) and the control law (8), (9) under Assumptions 1–3. For all initial conditions for which $u_0(x)$ is locally Lipschitz on $[0, D]$ and which satisfy the compatibility condition $u_0(D) = \kappa(p(D, 0))$, there exists a unique solution to the closed-loop system with $X(t) \in C^1[0, \infty)$ and $u(x, t)$ locally Lipschitz on $[0, D] \times [0, \infty)$. Moreover, there exists a class $\mathcal{KL}$ function Γ such that the following holds for all $t \geq 0$

$$|X(t)| + \sup_{x\in[0,D]} |u(x,t)| \leq \Gamma\left(|X(0)| + \sup_{x\in[0,D]} |u_0(x)|, t\right). \tag{10}$$

The Lipschitzness of the initial condition $u_0(x)$ and the compatibility condition guarantee that the closed-loop system is well posed.

The proof of Theorem 1 is based on the following lemmas whose proof can be found in Appendix (Section A).

Using the predictor state defined in (9), we introduce in the first two lemmas a novel backstepping transformation (and its inverse) that allows one to convert the original system to a suitable "target system," whose stability is easier to establish compared to the original closed-loop system (1)–(3), (8), (9).

Lemma 1: The control law defined in (8) and (9), together with the infinite-dimensional backstepping transformation

$$w(x,t) = u(x,t) - \kappa(p(x,t)) \tag{11}$$

where $p(x, t)$ is defined in (9), maps the system (1), (2) with the boundary condition (3) into the following target system:

$$\dot{X}(t) = f(X(t), \kappa(X(t)) + w(0,t)), \tag{12}$$

$$\partial_t w(x,t) = v(X(t))\partial_x w(x,t), \quad x \in [0, D] \tag{13}$$

$$w(D,t) = 0. \tag{14}$$

Lemma 2: The inverse of the infinite-dimensional backstepping transformation (11) is given by

$$u(x,t) = w(x,t) + \kappa(\pi(x,t)) \tag{15}$$

where π is defined as

$$\pi(x,t) = X(t) + \int_0^x \left[\frac{1}{v(\pi(y,t))} \times f\left(\pi(y,t), \kappa(\pi(y,t)) + w(y,t)\right)\right] dy. \tag{16}$$

In the next lemma we show that the target system (12)–(14) is globally asymptotically stable employing a Lyapunov-like argument.

Lemma 3: There exists a function $\nu \in \mathcal{KL}$, such that

$$|X(t)| + \|w(t)\|_\infty \leq \nu\left(|X(0)| + \|w(0)\|_\infty, t\right) \tag{17}$$

for all $t \geq 0$.

Lemmas 4 and 5 show the equivalence between the norm of the original system and the norm of the transformed system based on Assumptions 1–3.

Lemma 4: There exists a class $\mathcal{K}_\infty$ function $\bar{\omega}$, such that

$$\sup_{x\in[0,D]} |p(x,t)| \leq \bar{\omega}\left(|X(t)| + \sup_{x\in[0,D]} |u(x,t)|\right) \quad t \geq 0. \tag{18}$$

Lemma 5: There exists a class $\mathcal{K}_\infty$ function $\underline{\omega}$, such that

$$\sup_{x\in[0,D]} |\pi(x,t)| \leq \underline{\omega}\left(|X(t)| + \sup_{x\in[0,D]} |w(x,t)|\right), \quad t \geq 0. \tag{19}$$

Proof of Theorem 1: Assumption 3 implies the existence of a class $\mathcal{K}_\infty$ function Ω, such that

$$\kappa(|\xi|) \leq \Omega(|\xi|). \tag{20}$$

From the backstepping transformation (11) defined in Lemma 1 we deduce the following inequalities:

$$\sup_{x\in[0,D]} |w(x,t)| \leq \sup_{x\in[0,D]} \left(|u(x,t)| + \Omega(|p(x,t)|)\right), \tag{21}$$

$$\sup_{x\in[0,D]} |u(x,t)| \leq \sup_{x\in[0,D]} \left(|w(x,t)| + \Omega(|\pi(x,t)|)\right). \tag{22}$$

Then, from (18) and (19), we obtain

$$\sup_{x\in[0,D]} |w(x,t)| \leq \sup_{x\in[0,D]} |u(x,t)| + \Omega\circ\bar{\omega}\left(|X(t)| + \sup_{x\in[0,D]} |u(x,t)|\right), \tag{23}$$

$$\sup_{x\in[0,D]} |u(x,t)| \leq \sup_{x\in[0,D]} |w(x,t)| + \Omega\circ\underline{\omega}\left(|X(t)| + \sup_{x\in[0,D]} |w(x,t)|\right). \tag{24}$$

From (23) and (24), there exist some class $\mathcal{K}_\infty$ functions $\bar{\lambda}$ and $\underline{\lambda}$, such that

$$|X(t)| + \sup_{x\in[0,D]} |w(x,t)| \leq \bar{\lambda}\left(|X(t)| + \sup_{x\in[0,D]} |u(x,t)|\right), \tag{25}$$

$$|X(t)| + \sup_{x\in[0,D]} |u(x,t)| \leq \underline{\lambda}\left(|X(t)| + \sup_{x\in[0,D]} |w(x,t)|\right). \tag{26}$$

Combining (17) and (26), we conclude that

$$|X(t)| + \sup_{x\in[0,D]} |u(x,t)| \leq \underline{\lambda}\left(\nu\left(|X(0)| + \sup_{x\in[0,D]} |w_0(x)|, t\right)\right). \tag{27}$$

Using (25) we recover (10) with $\Gamma(s) = \underline{\lambda}\left(\nu(\bar{\lambda}(s))\right)$.

In order to prove the well-posedness of the closed-loop system consisting of (1)–(3) with the controller (8), (9), we first compute the solution to (13) and (14) with respect to a given initial condition $(X(0), w_0(x))$. We denote by $w(x(s), t(s))$ the characteristic curve passing through the point $(x,t) \in [0, D] \times [0, \infty)$, that is

$$\frac{dt(s)}{ds} = 1, \tag{28}$$

$$\frac{dx(s)}{ds} = -v(X(t(s))), \tag{29}$$

$$\frac{dw(s)}{ds} = 0 \tag{30}$$

with the initial conditions $t(0) = 0$, $x(0) = x_0$, and $w(0) = w_0(x_0)$, respectively. Integrating (28)–(30) along the characteristic lines, one deduces the solution of (13) and (14) as

$$w(x,t) = w_0\big(x + \Phi(t)\big), \quad \text{for all } 0 \le x + \Phi(t) \le D \tag{31}$$

$$w(x,t) = 0, \quad \text{for all } x + \Phi(t) \ge D, \tag{32}$$

$$\Phi(t) = \int_0^t v(X(\lambda))d\lambda. \tag{33}$$

Thus, for $t < \Phi^{-1}(D)$, system (12) is written as

$$\dot{X}(t) = f\left(X(t), \kappa\left(X(t)\right) + w_0\big(\Phi(t)\big)\right), \tag{34}$$

$$\dot{\Phi}(t) = v(X(t)), \tag{35}$$

$$\Phi(0) = 0. \tag{36}$$

From the backstepping transformation (11), we obtain

$$w_0(x) = u_0(x) - \kappa\left(p_0(x)\right) \tag{37}$$

where $p_0(x)$ is given by

$$p_0(x) = X(0) + \int_0^x \frac{1}{v(p_0(y))} f(p_0(y), u_0(y))dy. \tag{38}$$

Since κ, f, U, and v are continuously differentiable, we deduce the local Lipschitzness of $w_0(x)$ from the Lipschitzness of $u_0(x)$ stated in Theorem 1 and (38). Then, relations (34)–(36) imply the local Lipschitzness of the right-hand side of the (X, Φ) system, which in turn ensures the existence and uniqueness of $(X(t), \Phi(t)) \in C^1[0, \Phi^{-1}(D))$, where $\Phi^{-1}(D)$ satisfies

$$D = \int_0^{\Phi^{-1}(D)} v(X(\tau))d\tau. \tag{39}$$

For $t > \Phi^{-1}(D)$, $w(0,t) = 0$ and the dynamics of X in (34) are reduced to $\dot{X} = f(X, \kappa(X))$. The Lipschitzness of f and κ guarantee the existence and uniqueness of $X(t) \in C^1(\Phi^{-1}(D), \infty)$ and the compatibility condition guarantees that X is differentiable at $\Phi^{-1}(D)$, and thus, $X(t) \in C^1[0, \infty)$.

From (31), (32) and (13), (14) the well-posedness of X together with the continuous differentiability and the strict positiveness of the transport speed $v(X)$ imply the existence and uniqueness of $w(x,t)$ which is locally Lipschitz on (x,t), for all $(x,t) \in [0, D] \times [0, \infty)$. Using the equivalence between the signals $p(x,t)$ and $\pi(x,t)$ stated in (104), it can be deduced from (100) that the π-system satisfies the following PDE

$$\partial_t \pi(x,t) = v(X(t))\partial_x \pi(x,t), \tag{40}$$

$$\pi(0,t) = X(t). \tag{41}$$

Defining the characteristic curves parameterized by some variable τ and expressing the total derivative of $\pi(x(\tau), t(\tau))$ in order to derive the equivalent set of ODE for the system (40) along the characteristic lines, the solution of the transport PDE (40), compatible with the boundary condition (41) is written as[1]

$$\pi(x,t) = X\left(\Phi^{-1}\left(x + \Phi(t)\right)\right). \tag{42}$$

The existence and uniqueness of $(X(t), \Phi(t)) \in C^1[0, \infty)$ ensures that $\pi(x,t)$ is continuously differentiable on $[0, D] \times [0, \infty)$, and thus, from the inverse backstepping transformation (15) and the local Lipschitzness of $w(x,t)$ we get the local Lipschitzness of $u(x,t)$ on $[0, D] \times [0, \infty)$.

IV. Linking the PDE–ODE Cascade to an Implicit State-Dependent Input Delay System

In this section, we present an alternative state-dependent delay system representation of the transport PDE/nonlinear ODE cascade system (1)–(3). The method of characteristics is used first in order to solve the transport PDE equation (2). Defining the characteristic curves parameterized by some variable τ, the state of the PDE can be described by $u(x(\tau), t(\tau))$, whose total derivative is written as

$$\frac{du(x(\tau), t(\tau))}{d\tau} = \frac{\partial u}{\partial t}\frac{dt}{d\tau} + \frac{\partial u}{\partial \tau}\frac{dx}{d\tau}. \tag{43}$$

By comparing the total derivative and the transport equation (2), we deduce the following ODEs system:

$$\frac{dt(\tau)}{d\tau} = 1, \tag{44}$$

$$\frac{dx(\tau)}{d\tau} = -v\left(X(t(\tau))\right), \tag{45}$$

$$\frac{du(\tau)}{d\tau} = 0. \tag{46}$$

Integration of the ODEs (44) and (45) yields the characteristic curves of the PDE (2) given as

$$t(\tau) = t_0 + \tau, \quad t(0) = t_0 \tag{47}$$

$$x(\tau) = \int_0^\tau \frac{dx(\lambda)}{d\lambda}\, d\lambda + x(0) \tag{48}$$

$$= -\int_0^\tau v\left(X(t_0 + \lambda)\right) d\lambda + D, \quad x(0) = D. \tag{49}$$

Now, we define the primitive function of the variable transport velocity as

$$\Phi_X(t) = \int_0^t v\left(X(\lambda)\right) d\lambda. \tag{50}$$

Since the transport velocity v is assumed to be strictly positive, the function $\Phi_X(t)$ is a monotonically increasing function and

[1]The explicit derivation of such solutions is given in detail later on in Section IV and is similar to the procedure employed for the derivation of (31)–(33).

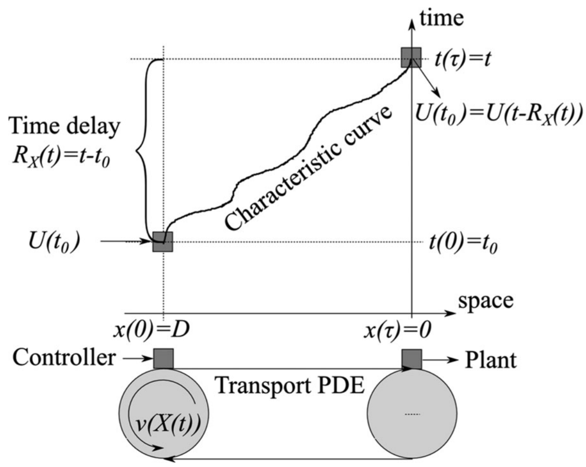

Fig. 2. Equivalence between the PDE/ODE cascade system and the delay system.

defines a bijective mapping between time and space. The subscript $_X$ denotes the state-dependence of the function $\Phi_X(t)$. By combining (49) and (50), we derive the following relation:

$$x(\tau) = \Phi_X(t_0) - \Phi_X(t_0 + \tau) + D. \tag{51}$$

We next reduce the PDE–ODE system to a state-dependent delay system (see Fig. 2). We consider the characteristic curves with $x(\tau) = 0$ at time $t = t_0 + \tau$ as illustrated in Fig. 2 and define the time delay $R_X(t) = t - t_0$. According to (51), the state-dependent input delay is implicitly given as

$$D = \Phi_X(t) - \Phi_X(t - R_X(t)). \tag{52}$$

Since the function $\Phi_X(t)$ depends on the state X of the plant, the delay $R_X(t)$ is also state-dependent. From (46), we know that the solution of the transport PDE (2) is constant along the characteristic curves. Thus

$$u(0,t) = u(D, t - R_X(t)) = U(t - R_X(t)). \tag{53}$$

Consequently, using (52) and (53), the original cascade system (1)–(3) is reduced to a nonlinear system with an implicit state-dependent input delay, which is written as

$$\dot{X}(t) = f\left(X(t), U(\phi(t))\right) \tag{54}$$

$$\phi(t) = t - R_X(t) \tag{55}$$

$$D = \int_{\phi(t)}^{t} v(X(\lambda))d\lambda. \tag{56}$$

We are not aware of a result dealing with the delay compensation of general nonlinear systems (54) with input delay of the form (55) and (56). A relevant result can be found in [9]. However, the results in [9] are dealing with linear ODE systems and the delay is defined implicitly through an integral of past input values rather than past state values [as in (56)]. In addition the result in [9] does not aim at completely compensating the input delay. A possible next step would be to consider the problem of delay compensation for general nonlinear ODE systems with input-dependent input delay of an integral type as the one considered in [9].

In the following, we design the predictor-feedback control law for the delay system (54)–(56) and present a stability analysis for the closed-loop system in delay system representation.

V. Predictor Feedback Control Design for the Equivalent Delay System

Let us define $\kappa(X)$ to be the nominal stabilizing feedback control law for the delay free plant $\dot{X}(t) = f\left(X(t), U(t)\right)$. The predictor feedback control law for system (54) is

$$U(t) = \kappa(P(t)) \tag{57}$$

where

$$P(t) = X(t) + \int_{\phi(t)}^{t} \frac{v(X(\theta))}{v(P(\theta))} f\left(P(\theta), U(\theta)\right) d\theta \tag{58}$$

with the initial condition

$$P(\theta) = X(0) + \int_{\phi(0)}^{\theta} \frac{v(X(s))}{v(P(s))} f\left(P(s), U(s)\right) ds \tag{59}$$

for all $\phi(0) \leq \theta \leq 0$. The fact that the predictor is given by (58) with the delay being defined by (56) can be seen as follows. Defining the prediction time

$$\sigma(t) = \phi^{-1}(t) \tag{60}$$

we derive the following implicit relation with respect to σ

$$D = \int_{t}^{\sigma(t)} v(X(\lambda))d\lambda. \tag{61}$$

Taking the time derivative of (61), we obtain

$$\dot{\sigma}(t) v\left(X(\sigma(t))\right) - v\left(X(t)\right) = 0 \tag{62}$$

that is

$$\dot{\sigma}(t) = \frac{v(X(t))}{v\left(X\left(\sigma(t)\right)\right)}. \tag{63}$$

Substitution of $t = \sigma(\theta)$, $\theta \in [\phi(t), t]$, in (54) leads to

$$\dot{X}(\sigma(\theta)) = \dot{\sigma}(\theta) f(X\left(\sigma(\theta)\right), U(\theta)). \tag{64}$$

Hence

$$\dot{X}(\sigma(\theta)) = \frac{v(X(\theta))}{v\left(X\left(\sigma(\theta)\right)\right)} f\left(X\left(\sigma(\theta)\right), U(\theta)\right). \tag{65}$$

Integrating (65) over $[\phi(t), t]$ and using definition

$$P(t) = X\left(\sigma(t)\right) \tag{66}$$

we derive the predictor (58) with initial condition (59).

To implement numerically the predictor feedback control law (57)–(59), one needs to compute at each time step $\phi(t)$ using (56) and employing the history of the state X. An example of computing numerically $\phi(t)$ is presented in the next section. Relevant numerical schemes for computation of a delay defined implicitly via an integral equation of the control input are presented in [9], [38]. Moreover, one then needs to numerically compute the predictor $P(t)$ using (58) employing, in addition, the history of the input U. We emphasize that in the recent papers [39], [40], the implementation issue of predictor feedback is discussed in detail and various numerical schemes are developed for computation of predictor feedback laws.

VI. Stability Analysis via Delay System Representation

Theorem 2: Consider the closed-loop system consisting of the plant (54)–(56) and the control law (57), (58) under Assumptions 1–3. For all initial conditions for which U and X are locally Lipschitz on the interval $[\phi(0), 0]$ and which satisfy the compatibility condition $U(0) = \kappa(P(0))$, there exists a unique solution to the closed-loop system with $X(t) \in C^1[0,\infty)$ and $U(t) \in C^1(0,\infty)$. Moreover, there exists a class $\mathcal{KL}$ function Λ, such that the following holds:

$$\Omega(t) \le \Lambda\left(\Omega(0), t\right), \quad t \ge 0 \tag{67}$$

where

$$\Omega(t) = \sup_{\phi(t)\le\theta\le t} |X(t)| + \sup_{\phi(t)\le\theta\le t} |U(\theta)|. \tag{68}$$

In order to prove Theorem 2 we state the following lemmas whose proofs are provided in Appendix (Section B).

Lemma 6: The infinite-dimensional backstepping transformation of the actuator state given by

$$W(\theta) = U(\theta) - \kappa(P(\theta)) \tag{69}$$

for all $\phi(t) \le \theta \le t$, together with the controller (57), (58) transform system (54)–(56) into the following target system:

$$\dot{X}(t) = f\left(X(t), \kappa(X(t)) + W(\phi(t))\right) \tag{70}$$

$$W(t) = 0. \tag{71}$$

Lemma 7: The inverse of the infinite-dimensional backstepping transformation (69) is defined for all $\phi(t) \le \theta \le t$ by

$$U(\theta) = W(\theta) + \kappa(\Pi(\theta)) \tag{72}$$

with

$$\Pi(\theta) = X(t) + \int_{t-R_X(t)}^{\theta} \left(\frac{v(X(\lambda))}{v(\Pi(\lambda)} \times f\left(\Pi(\lambda), \kappa(\Pi(\lambda)) + W(\lambda)\right)\right) d\lambda. \tag{73}$$

In the next lemma, we show that the target system (70), (71) is globally asymptotically stable constructing a Lyapunov functional. Note that the presented proof argument is different from the one used in the proof of Lemma 3.

Lemma 8: There exists a class $\mathcal{KL}$ function β, such that the following holds:

$$\Xi(t) \le \beta\left(\Xi(0), t\right), \quad t \ge 0 \tag{74}$$

$$\Xi(t) = \sup_{\phi(t)\le\theta\le t} |X(\theta)| + \sup_{\phi(t)\le\theta\le t} |W(\theta)|. \tag{75}$$

Lemma 9: There exists a class $\mathcal{K}_\infty$ function ρ such that the following holds for all $\phi(t) \le \theta \le t$

$$|P(\theta)| \le \rho\left(|X(t)| + \sup_{t-R_X(t)\le s\le t} |U(s)| \right). \tag{76}$$

Lemma 10: There exists a class $\mathcal{K}$ function ψ, such that the following holds:

$$|\Pi(\theta)| \le \psi\left(|X(t)| + \sup_{t-R_X(t)\le s\le t} |W(s)| \right) \tag{77}$$

for all $t - R_X(t) \le \theta \le t$.

Lemma 11: There exist class $\mathcal{K}_\infty$ functions ρ_1 and μ_4, such that the following holds:

$$\Omega(t) \le \mu_4(\Xi(t)), \tag{78}$$

$$\Xi(t) \le \rho_1(\Omega(t)) \tag{79}$$

where Ω and Ξ are defined in (67) and (75), respectively.

Proof of Theorem 2: Combining (78) and (79) with (74), we deduce that inequality (67) is satisfied with

$$\Lambda(s) = \mu_4^{-1}\left(\beta\left(\rho_1(s)\right), t\right). \tag{80}$$

We prove next, existence and uniqueness of solutions. We consider the system (X, ϕ) defined as

$$\dot{X}(t) = f\left(X(t), U(\phi(t))\right), \tag{81}$$

$$\dot{\phi}(t) = \frac{v(X(t))}{v(X(\phi(t))} \tag{82}$$

where the initial condition $\phi(0)$ satisfies the following relation:

$$D = \int_{\phi(0)}^{0} v(X(\lambda)) d\lambda. \tag{83}$$

For all $0 \le t < \sigma(0)$, it holds that $\phi(0) \le \phi(t) < 0$, and thus, since the initial conditions $X(s)$ and $U(s)$, $\phi(0) \le s < 0$, are Lipschitz the right-hand side of the (X, ϕ) system (81), (82) is Lipschitz with respect to (X, ϕ) . Thus, existence and uniqueness of $(X(t), \phi(t)) \in C^1[0, \sigma(0))$ follows.

Then, for $t > \sigma(0)$, from the target system $\dot{X} = f(X, \kappa(X))$ and the continuous differentiability of f and κ, we get existence and uniqueness of $X(t) \in C^1(\sigma(0), \infty)$. With the compatibility condition, we get that X is differentiable also at $\sigma(0)$, and thus, $X(t) \in C^1[0, \infty)$.

Differentiating (73) with respect to t, we have

$$\dot{\Pi}(t) = \frac{v(X(t))}{v\left(\Pi(t)\right)} f\left(\Pi(t)), \kappa(\Pi(t))\right), \quad \text{for all } t \ge 0. \tag{84}$$

Introducing the change of variables $\tau = \Phi_X(t)$ we rewrite system (84) as

$$\dot{\bar{\Pi}}(\tau) = \frac{1}{v\left(\bar{\Pi}(\tau)\right)} f\left(\bar{\Pi}(\tau)), \kappa(\bar{\Pi}(\tau))\right), \quad \text{for all } \tau \ge 0 \tag{85}$$

where $\bar{\Pi}(\tau) = \Pi\left(\Phi_X^{-1}(\tau)\right)$. Since κ, v, and f are continuously differentiable, it follows that there exists a unique solution $\bar{\Pi}(\tau) \in C^1[0, \infty)$. Thus, $\bar{\Pi}(\Phi_X(t))$ is continuously differentiable with respect to t for all $\Phi_X(t) \ge 0$, i.e., $t \ge \Phi_X^{-1}(0) = 0$, since $X(t) \in C^1[0, \infty)$ and $\dot{\Phi}_X(t) = v(X(t))$. Hence, since $\Pi(t) = \bar{\Pi}(\Phi_X(t))$, we deduce that $\Pi(t) \in C^1[0, \infty)$, and thus, we get that $U(t) \in C^1(0, \infty)$, which concludes the proof.

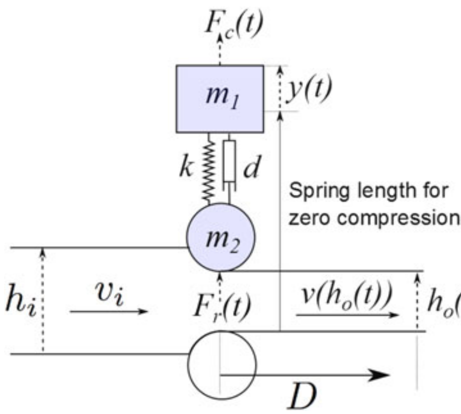

Fig. 3. Metal rolling schematic.

VII. Application to Metal Rolling Processes

The metal rolling process is a common industrial process, where, in essence, a deformation of a workpiece takes place between two rolls with parallel axes revolving in opposite directions as shown in Fig. 3 [41]. In industry, the initial breakdown of ingots is generally performed using hot rolling, while the cold rolling is crucial for the production of sheet or strip with good surface finishes and increased mechanical strength. However, in practice, often undesired self-excited vibrations occur, which are known as chatter [32] and are closely related to the machine tool vibrations in metal cutting [2]. In general, the reason for chatter vibrations are the interaction between the structural dynamics of the mill stand and the rolling process, where for unstable situations, energy from the machine drives is captured by the process and transformed into vibration energy of the structure. In these systems, time delays occur due to the material transport between two passes or between two stands of the mill [32], [42]. These delays are state-dependent due to their dependency on the state-dependent velocity of the metal strip [33], [43], but these are often approximated by constant delays [44]. There are several strategies to control the strip thickness in metal rolling [30], [31], [41], [45], but the effect of state-dependent delays on the dynamics of metal rolling is only rarely studied in the literature [34]. In this section, the compensation of the state-dependent delay in metal rolling via the predictor feedback control design from Section V is illustrated. In an industrial application of cold rolling the control task is very complex, including, for example, the control of interstand tension and interstand strip thickness between multiple mill stands as well as eccentricity control of the rolls. In the present contribution, we consider only strip thickness control at a single mill stand to focus on the compensation of the state-dependent delays, which are generic for these type of processes.

A. Modeling of Metal Rolling

The physical layout of the mill stand is illustrated in Fig. 3. It is closely related to the rolling model from [30], [31], and [41]. The bottom roll is assumed to be rigid, whereas the position of the upper roll is adjustable. The flexible roll with lumped mass m_2 is connected via a spring with stiffness k to a roll gap adjusting mechanism with lumped mass m_1. Both ends of the spring are movable and the equation of motion can be given by

$$m_1\ddot{y}(t) + d\big(\dot{y}(t) - \dot{h}_o(t)\big) + k\big(y(t) - h_o(t)\big) = F_c$$
$$m_2\ddot{h}_o(t) + d\big(\dot{h}_o(t) - \dot{y}(t)\big) + k\big(h_o(t) - y(t)\big) = F_r \quad (86)$$

where $y(t)$ and $h_o(t)$ specify the positions of the upper and the lower end of the spring, respectively. In particular, $h_o(t)$ is equivalent to the roll gap and the upper position $y(t)$ is defined in such a way that it is equivalent to the roll gap if the spring is not compressed. In contrast to [30], [31], and [41], we do not neglect the inertial force of the roll ($m_2 > 0$) and we consider a damping term with damping coefficient d. We assume that damping is proportional to the derivative of the relative displacement of the spring. The variables $F_c(t)$ and $F_r(t)$ denote the control and the process forces that act on the upper and the lower end of the spring, respectively. The process force F_r in metal rolling depends in a nonlinear way on the difference between the input thickness h_i and the output thickness $h_o(t)$ of the metal

$$F_r = F(h_o(t)) = k_f\sqrt{h_i - h_o(t)} \quad (87)$$

where the input thickness h_i is assumed to be constant and k_f specifies the force coefficient. Details on the derivation of the force law and the determination of the force coefficient k_f in metal rolling can be found in [32], [42], [45], and [46].

A feedback controller is used to keep the output thickness $h_o(t)$ at a desired reference value h_r by controlling the force F_c on the upper end of the spring. A PD controller is considered for the stabilization of the output thickness. In particular, the nominal controller is given by

$$F_c(t) = U(t) = K_p\big(h_r - h_o(t)\big) - K_d\dot{h}_o(t) - F_0 \quad (88)$$

where K_p and K_d are the gains for the proportional and the derivative terms, respectively. The constant part $F_0 = k_f\sqrt{h_i - h_r}$ is necessary to provide the constant rolling force for keeping the output thickness at the desired reference value h_r. In practice, the measurement point for the output strip thickness is located a constant distance D away from the rolling mill. In an industrial mill stand, realistic values for D range from 1 to 2 m, which implies that the delay cannot be neglected and significant delay variations are possible for lower speeds [44]. Hence, we assume that only a delayed version $U(\phi(t))$ of the feedback control input (88) can affect the plant. The delayed time $\phi(t)$ is given by (56), where $v(X(t))$ specifies the velocity of the metal strip over the constant distance D [33], [34], [43]. Due to mass conservation, the velocity $v(X(t))$ can be specified by [32], [43], [45]

$$v(h_o(t)) = \frac{h_i v_i}{h_o(t)} \quad (89)$$

where the input velocity v_i of the metal is constant and where it is assumed that the width of the strip does not change during the process [43]. Thus, a state-dependent delay $R_X(t)$ appears [33], [34], [43]. We only consider the case $h_o > 0$ because (89) is not adequate to model a collapse with $h_o = 0$.

We now introduce the state variables of the system as follows:

$$X_1(t) = h_0(t), X_2(t) = \dot{h}_0(t), X_3(t) = y(t), X_4(t) = \dot{y}(t). \tag{90}$$

The rolling example (86) with the force (87) and the state-dependent transport velocity (89) can be written as a system of delay differential equations (DDEs) with state-dependent delay, in the form (54)–(56), as

$$\begin{aligned}
\dot{X}_1(t) &= X_2(t),\\
\dot{X}_2(t) &= \frac{k_f}{m_2}\sqrt{h_i - X_1(t)} + \frac{d}{m_2}\big(X_4(t) - X_2(t)\big)\\
&\quad + \frac{k}{m_2}\big(X_3(t) - X_1(t)\big),\\
\dot{X}_3(t) &= X_4(t),\\
\dot{X}_4(t) &= \frac{d}{m_1}\big(X_2(t) - X_4(t)\big) + \frac{k}{m_1}\big(X_1(t) - X_3(t)\big)\\
&\quad + \frac{1}{m_1}U(\phi(t)),\\
D &= \int_{\phi(t)}^{t} \frac{h_i v_i}{X_1(\theta)} d\theta
\end{aligned} \tag{91}$$

where we have dropped all trivial time dependences t. Under the proposed control law (88) and realistic parameter values (namely, $h_i - h_r > 0$), the closed-loop system (91) has one equilibrium $X^* = (h_r, 0, y^*, 0)$, where $U = -F_0$ and

$$y^* = h_r - \frac{k_f}{k}\sqrt{h_i - h_r}. \tag{92}$$

The objective is to stabilize the equilibrium X^*.

B. Delay-Free Closed-Loop System

The predictor feedback control compensates the state-dependent delay $R_X(t) = t - \phi(t)$. Thus, ideally, the performance of the closed-loop system with delay under the predictor feedback control law would be equivalent to the performance of the closed-loop system without delay, and under the nominal delay-free feedback law, after a finite transient period (i.e., after the control signal reaches the plant). We briefly discuss the stability of the linearized delay-free closed-loop system. The characteristic equation of system (91) (linearized around the equilibrium X^*) without delay under the nominal control law (88) is given by

$$\begin{aligned}
m_1 m_2 s^4 &+ d(m_1 + m_2)s^3\\
&+ (k_r m_1 + k m_1 + k m_2 + dK_d)s^2\\
&+ (K_p d + K_d k + k_r d)s + k(K_p + k_r) = 0
\end{aligned} \tag{93}$$

where $k_r = k_f/(2\sqrt{h_i - h_r})$ is the stiffness of the rolling process at the equilibrium. For the open-loop system, that is, $K_p = 0$ and $K_d = 0$, it can be derived from the Routh–Hurwitz criterion that the equilibrium is always stable in the case of a physically meaningful choice of the parameters (all parameter values larger than zero and $h_i > h_r$). However, numerical simulations have shown that the open-loop system is only weakly stable, which means that potential transient oscillations decay very slowly. The maximum real part of the characteristic roots is calculated numerically from (93) for the plant parameter values from Table I and varying control parameters K_p and K_d. The result is presented in the contour plot in Fig. 4. In particular, for the open-loop system, the real part of the dominant characteristic root is given by $-0.0035\ \mathrm{s}^{-1}$, which means that a controller should be applied to suppress undesired long-lasting transient oscillations. The choice $K_p = 10^6$ N·m^{-1}, $K_d = 6500\ \mathrm{N\ s\ m}^{-1}$ (labeled by "x" in Fig. 4) leads to good nominal transient performance in the sense that the real part of the characteristic root is given by $-22.33\ \mathrm{s}^{-1}$.

TABLE I
PHYSICAL DEFINITION OF THE PARAMETERS

Symbol and value	Units (S.I)	Definition
$m_1 = 50$	kg	Mass of adjusting mechanism
$m_2 = 100$	kg	Mass of flexible roll
$d = 3000$	N sm^{-1}	Damping coefficient
$k = 10 \times 10^6$	N·m^{-1}	Stiffness
$K_F = 50000$	N·m$^{-0.5}$	Force constant
$h_i = 0.005$	m	Input thickness
$h_r = 0.003$	m	Desired output thickness
$v_i = 5$	m.s^{-1}	Input velocity
$D = 0.1$	m	Distance between rolls and sensor
$K_p = 1 \times 10^6$	N·m^{-1}	Proportional gain of PD-Controller
$K_d = 6500$	N sm^{-1}	Derivative gain of PD-Controller

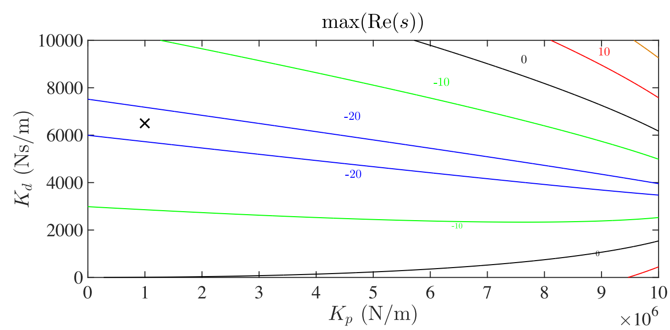

Fig. 4. Performance of the nominal control law for the system without delay.

C. Simulation Results for System With Delay

Simulations of the rolling process were performed by integrating the equations of motion (91) with the MATLAB solver *ddesd* for DDEs with state-dependent delay. The MATLAB solver requires an explicit expression for the delayed time $\phi(t)$, which can be obtained by numerical integration of the time derivative of (56)

$$\dot{\phi}(t) = \frac{v(X(t))}{v\left(X\left(\phi(t)\right)\right)} \tag{94}$$

with initial condition $\phi(0)$ obtained by numerically integrating (56) in the interval of the initial function. The predictor state $P(t)$ is obtained by a numerical integration of (58). We

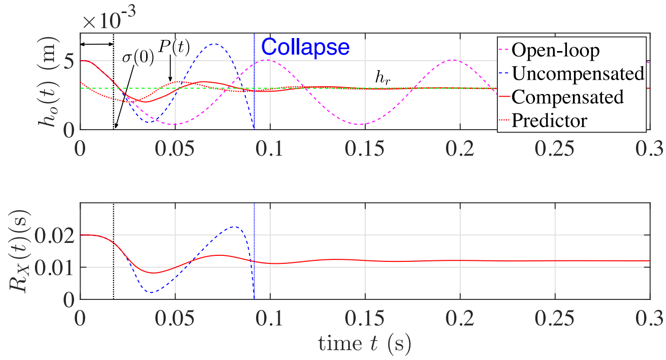

Fig. 5. Delay compensation: Upper position and implicit state-dependent delay evolution in time.

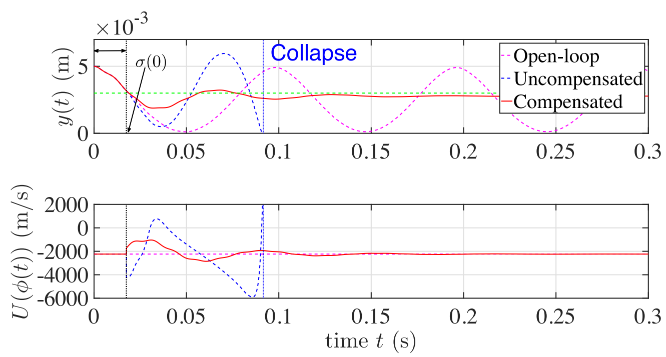

Fig. 6. Delay compensation: Strip thickness and input evolution in time.

have used constant upper position $y(\theta) = h_i$ and constant output thickness $h_o(\theta) = h_i$ with $\phi(0) \leq \theta \leq 0$ as initial conditions, which means that for $t < 0$ the rolling force and the control force are zero. At $t = 0$ the first input, which is given by the constant initial condition $U(\theta) = -F_0$, $\phi(0) \leq \theta \leq 0$, arrives at the plant and the deformation of material starts. The initial condition of the predictor is obtained by numerical integration of (59).

Three different cases were studied. On one hand, the uncompensated PD controller (88) with the nominal feedback law $\kappa(X(t))$ from (88) and the open-loop control law were employed. On the other hand, the predictor feedback law (57) for the nominal feedback law (88) was employed. The simulation reveals the necessity to compensate the delay in order to avoid collapse phenomena as it is shown in Figs. 5 and 6. More precisely, the uncompensated control action leads to a negative thickness, which is not admissible physically for the metal rolling dynamics. The simulation results also exhibit the limitation of the open-loop control with $F_c = -F_0$, which is not able to drive the system to the reference thickness value h_r fast enough as illustrated in Figs. 5 and 6. In addition, we have employed the nominal PD controller (88) in system (91) without delay ($D = 0$) to verify the equivalence between the response of the closed-loop system with delay ($D > 0$) under the predictor feedback control law and the response of the delay-free system with the nominal control law $D = 0$. Fig. 7 shows that, in fact, practically no deviations occur between the response of

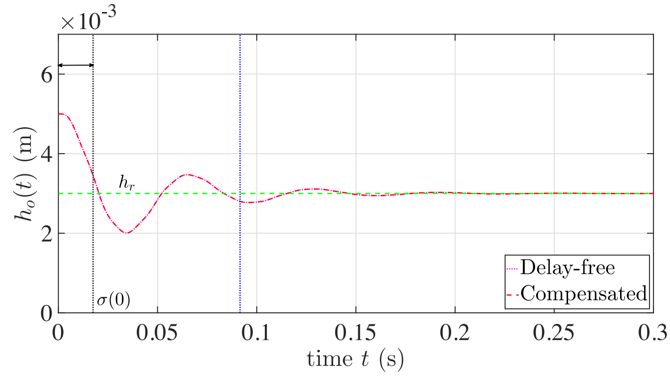

Fig. 7. Delay compensation: Delay-free and compensated plants evolution in time.

the nominal controller of the delay-free system and the response of the predictor feedback control in the delayed system. Note that for the delay-free system, we have used $U(t) = -F_0$ for the time interval $0 \leq \theta \leq \sigma(0)$ similar to the initial condition $U(t)$, $\phi(0) \leq t \leq 0$, for the input of the system with delay.

VIII. Conclusion

In this paper, we present the predictor feedback control design for transport PDE/nonlinear ODE cascades in which the transport coefficient depends on the ODE state. The proof of stability of the closed-loop system is established by using a backstepping transformation which maps the original system into a suitable target system whose stability is proven using a Lyapunov-like argument. The equivalence between the stability of the target and the original systems is stated using the invertibility of the backstepping transformation. An alternative representation of the coupled PDE–ODE system with a nonlinear system with state-dependent input delay is presented. The equivalent predictor-feedback control design for the delay system is introduced and an alternative proof of global asymptotic stability of the closed-loop system is provided constructing a Lyapunov functional. Consistent simulation results are provided applying the proposed algorithm to a model of a metal rolling process in which the control of the output thickness is a critical issue.

Appendix A
PDE Representation Lemmas' Proofs

A. Proof of Lemma 1

The proof of Lemma 1 is established in the following steps.

1) Differentiating (9) with respect to t, the following relation is deduced:

$$\begin{aligned}\partial_t p(x,t) = & -\int_0^x \frac{1}{v(p(y,t))} \\ & \times \frac{\nabla v(p(y,t))}{v(p(y,t))} [f(p(y,t),u(y,t))\partial_t p(y,t) \\ & - \partial_p f(p(y,t),u(y,t))\partial_t p(y,t)]\, dy\end{aligned}$$

$$+\int_0^x \frac{1}{v(p(y,t))} \times \partial_u f(p(y,t),u(y,t))\partial_t u(y,t)dy + f(p(0,t),u(0,t)). \quad (95)$$

Next, differentiating (9) with respect to x, we arrive at

$$v(X(t))\partial_x p(x,t) = -\int_0^x \left(v(X(t))\frac{\nabla v(p(y,t))}{v^2(p(y,t))} \times f(p(y,t),u(y,t))\partial_y p(y,t)\right)dy + \int_0^x \frac{v(X(t))}{v(p(y,t))} \times \partial_p f(p(y,t),u(y,t))\partial_y p(y,t)dy + \int_0^x \frac{v(X(t))}{v(p(y,t))} \times \partial_u f(p(y,t),u(y,t))\partial_y u(y,t)dy + \frac{v(X(t))}{v(p(0,t))}f(p(0,t),u(0,t)). \quad (96)$$

Combining (95) and (96), the following equality holds:

$$\partial_t p(x,t) - v(X(t))\partial_x p(x,t) = -\int_0^x \frac{1}{v(p(y,t))} \times \left[f(p(y,t),u(y,t))\frac{\nabla v(p(y,t))}{v(p(y,t))} \times \left(\partial_t p(y,t) - v(X(t))\partial_y p(y,t)\right)\right]dy + \int_0^x \frac{1}{v(p(y,t))}\partial_p f(p(y,t),u(y,t)) \times \left(\partial_t p(y,t) - v(X(t))\partial_y p(y,t)\right)dy. \quad (97)$$

Now, we define the function $G(x,t) = \partial_t p(x,t) - v(X(t))\partial_x p(x,t)$, which satisfies

$$\frac{dG(x,t)}{dx} = -\frac{1}{v(p(x,t))} \times \left[f(p(x,t),u(x,t))\frac{\nabla v(p(x,t))}{v(p(x,t))} - \partial_p f(p(x,t),u(x,t))\right]G(x,t) \quad (98)$$

$$G(0,t) = 0. \quad (99)$$

Hence, $G(x,t) = 0$ for all $x \in [0,D]$, which implies that

$$\partial_t p(x,t) - v(X(t))\partial_x p(x,t) = 0. \quad (100)$$

2) Taking the time and the spatial derivative of the backstepping transformation (11), we obtain

$$\partial_t w(x,t) = \partial_t u(x,t) - \partial_p \kappa\left(p(x,t)\right)\partial_t p(x,t) \quad (101)$$

and

$$\partial_x w(x,t) = \partial_x u(x,t) - \partial_p \kappa\left(p(x,t)\right)\partial_x p(x,t) \quad (102)$$

respectively, which leads to

$$\partial_t w(x,t) - v(X(t))\partial_x w(x,t) = -\partial_p \kappa\left(p(x,t)\right) \times \left[\partial_t p(x,t) - v(X(t))\partial_x p(x,t)\right] + \partial_t u(x,t) - v(X(t))\partial_x u(x,t). \quad (103)$$

Using (2) and (100), we derive (13) from (103). The ODE dynamics (12) and the boundary condition (14) are obtained by direct verification from (11) for $x = 0$ and (8), respectively.

B. Proof of Lemma 2

The inverse transformation (15) maps $w \mapsto u$ and is associated to the target system predictor, namely, (16) whereas the direct transformation (11), which maps $u \mapsto w$, is associated to the plant predictor, namely, (9). Thus, even if the two predictor representations are driven by different input signals, it holds that

$$p(x,t) = \pi(x,t), \quad \forall x \in [0,D]. \quad (104)$$

C. Proof of Lemma 3

Consider the following family of parameterized Lyapunov functions candidates for the target system's transport PDE (14)

$$L_{c,n}(t) = \int_0^D e^{2ncx}w^{2n}(x,t)dx \quad (105)$$

for any $c > 0$ and positive integer n. The time derivative of $L_{c,n}(t)$ along (13) and (14) is written as

$$\begin{aligned}\dot{L}_{c,n}(t) &= \int_0^D e^{2ncx}\partial_t w(x,t)^{2n}dx,\\ &= 2nv\left(X(t)\right)\int_0^D e^{2ncx}w(x,t)^{2n-1}\partial_x w(x,t)dx\\ &= -v(X(t))\left[w(0,t)^{2n} + 2nc\int_0^D e^{2ncx}w(x,t)^{2n}dx\right].\end{aligned} \quad (106)$$

From Assumption 1, it holds that $v(X(t)) \geq v_\star$, for all $X \in \mathbb{R}$, and thus, we obtain from (105) and (106) that

$$\dot{L}_{c,n}(t) \leq -2ncv_\star L_{c,n}(t). \quad (107)$$

Moreover, from (105) it follows that

$$\int_0^D |w(z,t)|^{2n}dz \leq L_{c,n}(t) \leq e^{2ncD}\int_0^D |w(z,t)|^{2n}dz \quad (108)$$

for all $t \geq 0$, $c > 0$, and $n \in \mathbb{N}_+$. Integrating (107) and using (108), we obtain

$$\int_0^D |w(z,t)|^{2n}dz \leq e^{-2ncv_\star(t-s)}e^{2ncD}\int_0^D |w(z,s)|^{2n}dz \quad (109)$$

for all $t \geq 0$, $s \geq 0$. From (109), we obtain

$$\left(\int_0^D |w(z,t)|^{2n} dz\right)^{\frac{1}{2n}} \leq e^{-cv_\star(t-s)} e^{cD} \times \left(\int_0^D |w(z,s)|^{2n} dz\right)^{\frac{1}{2n}}. \tag{110}$$

Taking the limit as $n \to \infty$ and using the fact that

$$\lim_{n\to\infty} \left(\int_0^D |w(z,t)|^{2n} dz\right)^{\frac{1}{2n}} = \sup_{x\in[0,D]} |w(x,t)| \equiv \|w(t)\|_\infty \tag{111}$$

from (110) the following holds:

$$\sup_{x\in[0,D]} |w(x,t)| \leq e^{-cv_\star(t-s)} e^{cD} \left(\sup_{x\in[0,D]} |w(x,s)|\right) \tag{112}$$

for all $t \geq s \geq 0$. Based on Assumption 3, there exist some $\bar{\nu} \in \mathcal{KL}$ and $\bar{\alpha} \in \mathcal{K}_\infty$, such that the solutions of (12) satisfy

$$|X(t)| \leq \bar{\nu}\big(|X(s)|, t-s\big) + \bar{\alpha}\left(\sup_{\tau\in[s,t]} |w(0,\tau)|\right) \tag{113}$$

for all $t \geq s \geq 0$. We perform the change of variables $s = \frac{t}{2}$ and rewrite (113) as

$$|X(t)| \leq \bar{\nu}\left(\left|X\left(\frac{t}{2}\right)\right|, \frac{t}{2}\right) + \bar{\alpha}\left(\sup_{\tau\in\left[\frac{t}{2},t\right]} |w(0,\tau)|\right). \tag{114}$$

The estimate of $\left|X\left(\frac{t}{2}\right)\right|$ follows by setting $s = 0$ and substituting t by $\frac{t}{2}$ into (113). Hence, the following holds:

$$\left|X\left(\frac{t}{2}\right)\right| \leq \bar{\nu}\left(|X(0)|, \frac{t}{2}\right) + \bar{\alpha}\left(\sup_{\tau\in\left[0,\frac{t}{2}\right]} |w(0,\tau)|\right). \tag{115}$$

From (112), we derive the estimates

$$\sup_{\tau\in\left[0,\frac{t}{2}\right]} \|w(\tau)\|_\infty \leq e^{cD} \sup_{x\in[0,D]} |w_0(x)|, \tag{116}$$

$$\sup_{\tau\in\left[\frac{t}{2},t\right]} \|w(\tau)\|_\infty \leq e^{-\frac{cv_\star}{2}t} e^{cD} \sup_{x\in[0,D]} |w_0(x)|. \tag{117}$$

Substituting (115)–(117) into (114) and using the fact that

$$|w(0,\tau)| \leq \sup_{x\in[0,D]} |w(x,\tau)| \tag{118}$$

leads to (17) with

$$\nu(r,s) = \bar{\nu}\left(\bar{\nu}\left(r, \frac{s}{2}\right) + \bar{\alpha}\left(re^{cD}\right), \frac{s}{2}\right) + \bar{\alpha}\left(e^{-\frac{cv_\star}{2}s} re^{cD}\right) + e^{-cv_\star s} re^{cD}. \tag{119}$$

D. Proof of Lemma 4

Taking the derivative of (9) with respect to x, we obtain

$$\partial_x p(x,t) = \frac{1}{v(p(x,t))} f(p(x,t), u(x,t)) \tag{120}$$

with the boundary condition

$$p(0,t) = X(t). \tag{121}$$

Now, considering that $\frac{1}{v(p(x,t))} > 0$, we obtain the following relation with the help of (7):

$$\frac{\partial C(p(x,t))}{\partial p} \frac{1}{v(p(x,t))} f(p(x,t), u(x,t)) \leq \frac{1}{v(p(x,t))}\Big(C(p(x,t)) + \mu_3(|u(x,t)|)\Big). \tag{122}$$

Using (120), we arrive at

$$\frac{\partial C(p(x,t))}{\partial x} \leq \frac{1}{v(p(x,t))} C(p(x,t)) + \frac{1}{v(p(x,t))} \mu_3(|u(x,t)|). \tag{123}$$

With the help of (5), inequality (123) yields

$$\frac{\partial C(p(x,t))}{\partial x} \leq \frac{1}{v_\star} C(p(x,t)) + \frac{1}{v_\star} \mu_3(|u(x,t)|). \tag{124}$$

By the comparison principle and relation (121), we obtain

$$C(p(x,t)) \leq e^{\frac{x}{v_\star}} C(X(t)) + \frac{1}{v_\star} \int_0^x e^{\frac{x-y}{v_\star}} \mu_3(|u(y,t)|)\, dy \tag{125}$$

which leads to

$$C(p(x,t)) \leq e^{\frac{D}{v_\star}} C(X(t)) + \left(e^{\frac{D}{v_\star}} - 1\right) \mu_3\left(\sup_{x\in[0,D]} |u(x,t)|\right). \tag{126}$$

Using (6), the following inequality holds:

$$|p(x,t)| \leq \mu_1^{-1}\left(e^{\frac{D}{v_\star}} \mu_2(|X(t)|) + \left(e^{\frac{D}{v_\star}} - 1\right) \times \mu_3\left(\sup_{x\in[0,D]} |u(x,t)|\right)\right), \quad \text{for all} x \in [0,D]. \tag{127}$$

Defining

$$\bar{\omega}(s) = \mu_1^{-1}\left(e^{\frac{D}{v_\star}} \mu_2(s) + \left(e^{\frac{D}{v_\star}} - 1\right) \mu_3(s)\right) \tag{128}$$

the proof is complete.

E. Proof of Lemma 5

Differentiating (16), with respect to x, the following ODE is derived for all $x \in [0,D]$:

$$\partial_x \pi(x,t) = \frac{1}{v(\pi(x,t))} f\Big(\pi(x,t), \kappa(\pi(x,t)) + w(x,t)\Big), \tag{129}$$

$$\pi(0,t) = X(t). \tag{130}$$

We introduce next the following change of variables:

$$y(x,t) = t + \int_0^x \frac{dr}{v(\pi(r,t))}, \quad x \in [0,D] \tag{131}$$

where t acts as a parameter. Since the transport velocity v is assumed to be strictly positive, the function y is monotonically increasing with respect to x, for each t. Thus, it admits an inverse defined for each t as $x = \chi(y,t)$. Next, we rewrite the ODE (129), (130) as

$$\frac{1}{v(\pi(\chi(y,t),t))}\partial_y \pi(\chi(y,t),t) = \frac{1}{v(\pi(\chi(y,t),t))} f\Big(\pi(\chi(y,t),t), \kappa(\pi(\chi(y,t),t)) + w(\chi(y,t),t)\Big), \tag{132}$$

$$\pi(0,t) = X(t) \tag{133}$$

for all $y \in [t, t + \int_0^D \frac{dr}{v(\pi(r,t))}]$. Defining the change of variables

$$\psi(y,t) = \pi\Big(\chi(y,t),t\Big), \tag{134}$$

$$\omega(y,t) = w\Big(\chi(y,t),t\Big) \tag{135}$$

we rewrite (129) and (130) in the new coordinates as

$$\partial_y \psi(y,t) = f\Big(\psi(y,t), \kappa(\psi(y,t)) + \omega(y,t)\Big), \tag{136}$$

$$\psi(t,t) = X(t) \tag{137}$$

for all $y \in [t, t + \int_0^D \frac{dr}{v(\pi(r,t))}]$. Under Assumption 3, from (136), we deduce the existence of a class $\mathcal{K}_\infty$ function ν_3 and a class $\mathcal{K}$ function μ_6, such that

$$|\psi(y,t)| \leq \nu_3\left(|X(t)|, \int_0^D \frac{dr}{v(\pi(r,t))}\right) + \mu_6\left(\sup_{y\in\left[t,t+\int_0^D \frac{dr}{v(\pi(r,t))}\right]} |\omega(y,t)|\right),$$
$$\text{for all} y \in \left[t, t + \int_0^D \frac{dr}{v(\pi(r,t))}\right]. \tag{138}$$

Then, with the help of (131)–(135), the following inequality holds:

$$|\pi(x,t)| \leq \nu_3\left(|X(t)|, \int_0^D \frac{dr}{v(\pi(r,t))}\right) + \mu_6\left(\sup_{x\in[0,D]} |\omega(x,t)|\right) \tag{139}$$

for all $x \in [0,D]$. Knowing that ν_3 decreases with respect to its second argument, using the fact that $y(D,t) - t = \int_0^D \frac{dr}{v(\pi(r,t))} \geq 0$, the following holds:

$$\sup_{x\in[0,D]} |\pi(x,t)| \leq \nu_3\Big(|X(t)|, 0\Big) + \mu_6\left(\sup_{x\in[0,D]} |\omega(x,t)|\right). \tag{140}$$

Finally, using the properties of class $\mathcal{K}_\infty$ and $\mathcal{KL}$ functions, we obtain the inequality (19).

Appendix B
Delay System Representation Lemmas' Proofs

A. Proof of Lemma 6

The proof of Lemma 6 is based on a direct verification considering that $P(\phi(t)) = X(t)$.

B. Proof of Lemma 7

By direct verification considering that $P(\theta) \equiv \Pi(\theta)$ for all $\phi(t) \leq \theta \leq t$.

C. Proof of Lemma 8

Based on the input-to-state stability of $\dot{X} = f(X, \kappa(X) + \omega)$ with respect to ω, namely, Assumption 3, from [47], there exist a smooth function $S(X) : \mathbb{R}^n \to \mathbb{R}_+$ and class $\mathcal{K}_\infty$ functions $\alpha_1, \alpha_2, \alpha_3$, such that for any $\mu > 0$

$$\alpha_1(|X|) \leq S(X) \leq \alpha_2(|X|), \tag{141}$$

$$\frac{\partial S(X)}{\partial X} f(X, \kappa(X) + \omega)) \leq -\mu S(X) + \alpha_3(|\omega|). \tag{142}$$

Define next for any $c > 0$ and any positive integer n the functional

$$\bar{L}_{c,n}(t) = \frac{1}{v_\star}\int_{\phi(t)}^t e^{2nc(\Phi_X(\theta)+D-\Phi_X(t))} W(\theta)^{2n} d\theta \tag{143}$$

where Φ_X is defined in (50).

Taking the derivative of (143) and using (71), we obtain

$$\dot{\bar{L}}_{c,n}(t) = -\frac{1}{v_\star}\frac{d\phi(t)}{dt} e^{2nc(\Phi_X(\phi(t))+D-\Phi_X(t))} W(\phi(t))^{2n} - \frac{2nc}{v_\star}\Phi'_X(t)\int_{\phi(t)}^t e^{2nc(\Phi_X(\theta)+D-\Phi_X(t))} W(\theta)^{2n} d\theta. \tag{144}$$

From the implicit definition of the delay in (56), we deduce the following equality:

$$\dot{\phi}(t) = 1 - \frac{dR_X(t)}{dt} = \frac{v(X(t))}{v(X(\phi(t))} \tag{145}$$

and thus, from Assumption 1, we get that $\dot{\phi}(t) > 0$, for all $t \geq 0$. Moreover, from (50), we obtain the following equality:

$$\Phi'_X(t) = v(X(t)). \tag{146}$$

Therefore, Assumption 1 enables one to state the following inequality:

$$\dot{\bar{L}}_{c,n}(t) \leq -2ncv_\star \bar{L}_{c,n}(t). \tag{147}$$

Let us now define for any $c > 0$ the functional

$$\bar{L}(t) = \frac{1}{v_\star}\int_{\phi(t)}^{t} e^{c(\Phi_X(\theta)+D-\Phi_X(t))}\gamma(|W(\theta)|)v(X(\theta))d\theta \tag{148}$$

for any class $\mathcal{K}_\infty$ function γ. The derivative of $\bar{L}$ with respect to time is written as

$$\begin{aligned}\dot{\bar{L}}(t) = &-\frac{1}{v_\star}\frac{d\phi(t)}{dt}\\ &\times e^{c(\Phi_X(\phi(t))+D-\Phi_X(t))}\gamma\left(|W(\phi(t)|\right)v(X(\phi(t)))\\ &-\frac{c}{v_\star}\Phi_X'(\phi(t))\\ &\times\int_{\phi(t)}^{t} e^{c(\Phi_X(\theta)+D-\Phi_X(t))}\gamma\left(|W(\theta)|\right)v(X(\theta))d\theta\end{aligned} \tag{149}$$

where we use (71). Inserting (146) into (149) and using (145), we arrive at

$$\begin{aligned}\dot{\bar{L}}(t) = &-\frac{v(X(t))}{v_\star}e^{c(\Phi_X(\phi(t))+D-\Phi_X(t))}\gamma\left(|W(\phi(t)|\right)\\ &-\frac{c}{v_\star}v(X(t))\int_{\phi(t)}^{t}\Big\{e^{c(\Phi_X(\theta)+D-\Phi_X(t))}\\ &\times\gamma\left(|W(\theta)|\right)v(X(\theta))\Big\}d\theta.\end{aligned} \tag{150}$$

From (5), (52), and (55), we obtain

$$\dot{\bar{L}}(t) \le -\gamma\left(|W(\phi(t)|\right) - cv_\star\bar{L}(t). \tag{151}$$

Moreover, defining the functional

$$V_1(t) = S(X(t)) + \bar{L}(t) \tag{152}$$

whose time derivative along (70) is written as

$$\dot{V}_1(t) = \frac{\partial S(X(t))}{\partial X}f\left(X(t),\kappa(X(t))+W(\phi(t))\right)+\dot{\bar{L}}(t) \tag{153}$$

and combining (142) with (143), from (151), we obtain the following inequality:

$$\begin{aligned}\dot{V}_1(t) \le &-\mu S(X(t)) - cv_\star\bar{L}(t) + \alpha_3(|W(\phi(t))|)\\ &-\gamma\left(|W(\phi(t))|\right).\end{aligned} \tag{154}$$

Choosing γ such that $\gamma(s)\ge\alpha_3(s)$, for all $s\ge 0$, we obtain

$$\dot{V}_1(t) \le -\lambda V_1(t) \tag{155}$$

where

$$\lambda = \min\{\mu, cv_\star\}. \tag{156}$$

Let us define the Lyapunov function for the target system (70) and (71) as

$$V_n(t) = V_1(t)^{2n} + \bar{L}_{c,n}(t). \tag{157}$$

Taking the derivative of V_n with the help of (149) and (155), we obtain

$$\dot{V}_n(t) \le -2n\lambda V_n(t). \tag{158}$$

Therefore

$$V_n(t)^{\frac{1}{2n}} \le e^{-\lambda t}V_n(0)^{\frac{1}{2n}}. \tag{159}$$

It then follows that

$$V_1(t) + \bar{L}_{c,n}(t)^{\frac{1}{2n}} \le 2e^{-\lambda t}\left(V_1(0)+\bar{L}_{c,n}(0)^{\frac{1}{2n}}\right). \tag{160}$$

From (143), the following holds:

$$\bar{L}_{c,n}(t)^{\frac{1}{2n}} = \frac{1}{v_\star^{\frac{1}{2n}}}\left(\int_{\phi(t)}^{t} e^{2nc(\Phi_X(\theta)+D-\Phi_X(t))}W(\theta)^{2n}d\theta\right)^{\frac{1}{2n}}. \tag{161}$$

Thus, taking the limit as $n\to\infty$ of (161) and using the fact that

$$\begin{aligned}\lim_{n\to\infty}\bar{L}_{c,n}(t)^{\frac{1}{2n}} &= \sup_{\phi(t)\le\theta\le t}\left|e^{c(\Phi_X(\theta)+D-\Phi_X(t))}W(\theta)\right|\\ &\equiv \|W(t)\|_{c,\infty}\end{aligned} \tag{162}$$

we conclude that the following holds:

$$V_1(t) + \|W(t)\|_{c,\infty} \le 2e^{-\lambda t}\Big(V_1(0)+\|W(0)\|_{c,\infty}\Big). \tag{163}$$

From Assumption 1 and (148), it follows that

$$\begin{aligned}\bar{L}(t) \le &\frac{1}{v_\star}\sup_{\phi(t)\le\theta\le t}\left|e^{c(\Phi_X(\theta)+D-\Phi_X(t))}\gamma(|W(\theta)|)\right|\\ &\times\int_{\phi(t)}^{t} v(X(\theta))d\theta.\end{aligned} \tag{164}$$

Using relation (56) and the definition of the supremum norm

$$\|W(t)\|_\infty = \sup_{\phi(t)\le\theta\le t}\left|W(\theta)\right| \tag{165}$$

with the fact that Φ_X is an increasing function [that follows from (50)], we deduce the following estimate:

$$\bar{L}(t) \le \frac{D}{v_\star}e^{cD}\gamma\left(\|W(t)\|_\infty\right). \tag{166}$$

From the definition of V_1 in (152), using the facts that

$$\|W(t)\|_\infty \le \|W(t)\|_{c,\infty} \le e^{cD}\|W(t)\|_\infty \tag{167}$$

and that $S(X(t)) \le V_1(t)$, together with (141) and (166), we obtain

$$\begin{aligned}\alpha_1(|X(t)|) + \|W(t)\|_\infty \le &\ 2e^{-\lambda t}\Big(\alpha_2(|X(0)|)\\ &+\ \frac{D}{v_\star}e^{cD}\gamma\left(\sup_{\phi(0)\le\theta\le 0}\left|W(\theta)\right|\right)\\ &+\ e^{cD}\sup_{\phi(0)\le\theta\le 0}\left|W(\theta)\right|\Big).\end{aligned} \tag{168}$$

With the properties of comparison functions and the fact that $|X(0)| \le \sup_{\phi(0)\le s\le 0}|X(s)|$, we conclude that there exists a class $\mathcal{KL}$ function β_2, such that

$$|X(t)| + \sup_{\phi(t)\le s\le t} |W(s)| \le \beta_2\Bigg(\sup_{\phi(0)\le s\le 0} |X(s)| + \sup_{\phi(0)\le s\le 0} |W(s)|, t\Bigg). \quad (169)$$

Next, we upper bound $\sup_{\phi(t)\le s\le t}|X(s)|$. From (5), we deduce

$$\dot{\sigma}(\theta) \le \frac{v(X(\theta))}{v_\star}. \quad (170)$$

Integrating (170) on $[\phi(t), \theta]$ with $\sigma(\phi(t)) = t$, we derive the inequality

$$\sigma(\theta) - t \le \frac{1}{v_\star}\int_{\phi(t)}^{\theta} v(X(\lambda))d\lambda \quad (171)$$

for all $\phi(t) \le \theta \le t$. Since $v(X(t))$ is a positive function, it follows that $\int_{\phi(t)}^{\theta} v(X(\lambda))d\lambda$ is an increasing function of θ. Using the implicit definition of the delay in (56), we obtain

$$\sigma(\theta) - t \le \frac{D}{v_\star}, \quad \forall\ \phi(t) \le \theta \le t. \quad (172)$$

From inequality (172), the following holds:

$$\sigma(0) \le \tfrac{D}{v_\star}. \quad (173)$$

Dividing the time domain into three different intervals the following estimates are then obtained.

1) For $0 \le t \le \sigma(0)$, we have that $\phi(0) \le \phi(t) \le 0$. Therefore

$$\begin{aligned}\sup_{\phi(t)\le s\le t}|X(s)| &\le \sup_{\phi(0)\le s\le 0}|X(s)| + \sup_{0\le s\le t}|X(s)|\\ &\le \sup_{\phi(0)\le s\le 0}|X(s)| + \beta_2\left(\sup_{\phi(0)\le s\le 0}|X(s)| + \sup_{\phi(0)\le s\le 0}|W(s)|, 0\right).\end{aligned} \quad (174)$$

Thus, there exists a class $\mathcal{K}_\infty$ function μ_5, such that

$$\sup_{\phi(t)\le s\le t}|X(s)| \le \mu_5\Bigg(\sup_{\phi(0)\le s\le 0}|X(s)| + \sup_{\phi(0)\le s\le 0}|W(s)|\Bigg). \quad (175)$$

2) For $\sigma(0) \le t \le \frac{D}{v_\star}$, we have $0 \le \phi(t) \le \phi\left(\frac{D}{v_\star}\right)$. Thus

$$\sup_{\phi(t)\le s\le t}|X(s)| \le \sup_{0\le s\le t}|X(s)| \quad (176)$$

$$\le \beta_2\Bigg(\sup_{\phi(0)\le s\le 0}|X(s)| + \sup_{\phi(0)\le s\le 0}|W(s)|, 0\Bigg). \quad (177)$$

(3) For $t \ge \frac{D}{v_\star}$, we have from (173) that $\phi(t) \ge \phi(\frac{D}{v_\star}) \ge 0$. Thus, using (169), we arrive at

$$\sup_{\phi(t)\le s\le t}|X(s)| \le \beta_2\Bigg(\sup_{\phi(0)\le s\le 0}|X(s)| + \sup_{\phi(0)\le s\le 0}|W(s)|, \phi(t)\Bigg) \quad (178)$$

and integrating (145) over $[t, \sigma(t)]$ with the help of (5) and (61), we get the following inequality:

$$R_X(t) \le \frac{D}{v_\star}, \quad \forall t \ge 0 \quad (179)$$

we deduce that $t - R_X(t) \ge t - \frac{D}{v_\star}$, which leads to the existence of a class $\mathcal{KL}$ function $\bar{\beta}_2$, such that

$$\sup_{\phi(t)\le s\le t}|X(s)| \le \bar{\beta}_2\Bigg(\sup_{\phi(0)\le s\le 0}|X(s)| + \sup_{\phi(0)\le s\le 0}|W(s)|, t - \frac{D}{v_\star}\Bigg). \quad (180)$$

Combining estimates (175), (177), and (180), we deduce the existence of a class $\mathcal{KL}$ function $\bar{\beta}$, such that for all $t \ge 0$

$$\sup_{\phi(t)\le s\le t}|X(s)| \le \bar{\beta}\Bigg(\sup_{\phi(0)\le s\le 0}|X(s)| + \sup_{\phi(0)\le s\le 0}|W(s)|, t\Bigg). \quad (181)$$

D. Proof of Lemma 9

Differentiating (58), we deduce the following relation for all $\phi(t) \le \theta \le t$:

$$\frac{dP(\theta)}{d\theta} = \frac{v(X(\theta))}{v(P(\theta))} f\left(P(\theta), U(\theta)\right) d\theta. \quad (182)$$

Introducing the change of variables $y = \sigma(\theta)$, (182) may be rewritten as

$$\frac{dP(\phi(y))}{dy} = f\Big(P(\phi(y)), U\left(\phi(y)\right)\Big), \quad (183)$$

$$t \le y \le \sigma(t).$$

From Assumption 2, the following holds:

$$\frac{\partial C(P(\phi(y)))}{\partial y} \le C(P(\phi(y))) + \mu_3\Big(\left|U\left(\phi(y)\right)\right|\Big). \quad (184)$$

Using the comparison principle and the facts that $P(\phi(t)) = X(t)$ and $y = \sigma(\theta)$, we derive the following inequality:

$$C(P(\theta)) \le e^{(\sigma(\theta)-t)}\Bigg(C(X(t)) + \sup_{t\le s\le \sigma(t)} \mu_3(|U(\phi(s))|)\Bigg) \quad (185)$$

for all $\phi(t) \le \theta \le t$. Imposing $\theta = t$ in (171), we obtain

$$\sigma(t) - t \le \frac{1}{v_\star}\int_{\phi(t)}^{t} v(X(\lambda))d\lambda. \quad (186)$$

Combining (186) with the implicit definition of the delay in (56) and using the fact that σ is increasing, we get from (185) that

$$C(P(\theta)) \leq e^{\frac{D}{v_\star}} \left(C(X(t)) + \sup_{\phi(t) \leq \theta \leq t} \mu_3(|U(s)|) \right), \quad (187)$$

$$\phi(t) \leq \theta \leq t.$$

With standard properties of class $\mathcal{K}$ functions and using (6), we get (76), where the class $\mathcal{K}_\infty$ function ρ is written as

$$\rho(s) = \mu_1^{-1} \left(\left(\mu_2(s) + s \right) e^{\frac{D}{v_\star}} \right). \quad (188)$$

E. Proof of Lemma 10

Consider the change of variables $y = \sigma(\theta)$ and write the predictor of the target system (73) as

$$\frac{d\Pi(\phi(y))}{dy} = f\left(\Pi(\phi(y)), \kappa(\Pi(\phi(y))) + W(\phi(y))\right),$$

$$t \leq y \leq \sigma(t). \quad (189)$$

Under Assumption 3, there exist a class $\mathcal{KL}$ function β_3 and a class $\mathcal{K}$ function ψ_1, such that

$$\Pi(\phi(y)) \leq \beta_3\left(|X(t)|, y - t\right) + \psi_1\left(\sup_{t \leq s \leq y} |W(\phi(s))| \right),$$

$$t \leq y \leq \sigma(t). \quad (190)$$

Using the fact that $y = \sigma(\theta)$, we obtain

$$|\Pi(\theta)| \leq \psi_2(|X(t)|) + \psi_1\left(\sup_{t - R_X(t) \leq s \leq t} |W(s)| \right) \quad (191)$$

for all $t - R_X(t) \leq \theta \leq t$ with $\psi_2(s) = \beta_3(s, 0)$. Using the properties of class $\mathcal{K}$ functions, (77) is deduced with $\psi(s) = \psi_1(s) + \psi_2(s)$.

F. Proof of Lemma 11

Due to the continuity of $\kappa(.)$ and the fact that $\kappa(0) = 0$, there exists $\hat{\rho} \in \mathcal{K}_\infty$, such that

$$|\kappa(\xi)| \leq \hat{\rho}(|\xi|). \quad (192)$$

Using (192), the inverse transformation (72), and the bound (77), we derive (78) with

$$\mu_4(s) = s + \hat{\rho}(\psi(s)). \quad (193)$$

From the direct transformation (69) and the bound (76), we deduce (79), where ρ_1 is defined as

$$\rho_1(s) = s + \hat{\rho}(\rho(s)). \quad (194)$$

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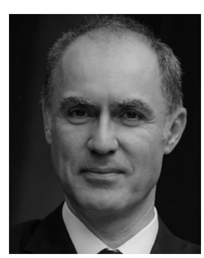

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