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Compensation of infinite-dimensional input dynamics☆

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Abstract: We present a tutorial introduction to methods for stabilization of systems with infinite-dimensional input dynamics including delays, diffusion, counter-convection and wave propagation. The methods are based on techniques originally developed for boundary control of partial differential equations. We consider multi-input linear time-invariant systems with input dynamics governed by distributed delays, and diffusion with counter-convection or wave PDEs. For the special case of single-input linear time-invariant systems with a single discrete delay we prove robustness of the control law to a small uncertainty in the delay and in the case of completely unknown delay we present an adaptive control approach. For this special case, we also present a method for compensating arbitrarily large but known time-varying delays. Finally, we consider nonlinear control problems in the presence of arbitrarily long input delays.

1. Introduction

A wealth of knowledge and research results exist for control of systems with state delays and input delays. Problems with long input delays, for unstable plants, represent a particular challenge. In fact, they were the first challenge to be dealt with, in Otto J. M. Smith's article (Smith, 1959), where a compensator, now known as the Smith predictor, was introduced five decades ago. The Smith predictor's value is in its ability to compensate for a long input or output delay in set point regulation or constant disturbance rejection problems. However, its major limitation is that, when the plant is unstable, it fails to recover the stabilizing property of a nominal controller for the plant without delay.

A substantial modification to the Smith predictor, which removes its limitation to stable plants was developed three decades ago in the form of finite spectrum assignment (FSA) controllers (Artstein, 1982; Kwon & Pearson, 1980; Manitius & Olbrot, 1979). More recent treatment of this subject can be found in the books (Michiels & Niculescu, 2007; Zhong, 2006a). In the FSA approach, the system

$$\dot{X}(t) = AX(t) + BU(t-D), \tag{1}$$

where X is the state vector, U is the control input (scalar in our consideration here), D is an arbitrarily long delay, and (A, B) is a controllable pair, is stabilized with the infinite-dimensional predictor feedback

$$U(t) = K\left[e^{AD}X(t) + \int_{t-D}^{t} e^{A(t-\theta)}BU(\theta)d\theta\right], \tag{2}$$

where the gain K is chosen so that the matrix $A + BK$ is Hurwitz. The word 'predictor' comes from the fact that the bracketed quantity is the future state $X(t + D)$, expressed using the current state $X(t)$ as the initial condition and using the controls $U(\theta)$ from the past time window $[t - D, t]$. Concerns are raised in (Mondie & Michiels, 2003) regarding the robustness of the feedback law (2) to digital implementation of the distributed delay (integral) term but are resolved with appropriate discretization schemes (Zhong, 2006b; Zhong & Mirkin, 2002).

One can view the feedback law (2) as implicit, since U appears both on the left and on the right. However, one should observe that the input memory $U(\theta)$, $\theta \in [t - D, t]$ is a part of the state of the overall infinite-dimensional system, so the control law is in fact given by an explicit full-state feedback formula. The predictor feedback (2) represents a particular form of boundary control, commonly encountered in control of partial differential equations.

Following our recent studies in solving boundary control problems for various classes of partial differential equations (PDEs) using the continuum version of the backstepping method (Krstic & Smyshlyaev, 2008; Vazquez & Krstic, 2007), we review in this article several extensions to the predictor feedback design that we have recently developed, particularly for nonlinear and PDE systems. These extensions, presented in the book (Krstic, 2009a), include the extension of predictor feedback to nonlinear systems and PDEs with input delays, robustness and inverse optimality results, a delay-adaptive design, an extension to time-

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varying delays, and observer design in the presence of sensor delays and PDE dynamics. Moreover, combining the PDE backstepping approach with a recently developed infinite-dimensional forwarding transformation, we design control laws for MIMO LTI systems with distributed input dynamics governed by diffusion with counter-convection or wave PDEs (see Bekiaris-Liberis & Krstic, 2010a and Bekiaris-Liberis & Krstic, 2010b).

2. Lyapunov functional and its benefits

2.1. Single-input systems with discrete delay

The key to extensions of the predictor feedback that we present here is the observation that the invertible backstepping transformation

$$w(x,t) = u(x,t) - \int_0^x K e^{A(x-y)} B u(y,t) dy - K e^{Ax} X(t), \tag{3}$$

$$u(x,t) = w(x,t) + \int_0^x K e^{(A+BK)(x-y)} \, B w(y,t) \, dy + K e^{(A+BK)x} X(t), \tag{4}$$

where

$$u(x,t) = U(t+x-D), \tag{5}$$

can transform the system (1), (2) into the *target system*

$$\dot{X}(t) = (A+BK)X(t) + Bw(0,t), \tag{6}$$

$$w_t(x,t) = w_x(x,t), \tag{7}$$

$$w(D,t) = 0, \tag{8}$$

which is a cascade of an undriven transport PDE w-subsystem and the exponentially stable X-system. Since an undriven transport PDE is exponentially stable, the overall cascade is exponentially stable. This fact is established with a Lyapunov functional

$$V(t) = X(t)^{\mathrm{T}} P X(t) + 2 \frac{|PB|^2}{\lambda_{\min}(Q)} \int_0^D (1+x) w(x,t)^2 \, dx, \tag{9}$$

where P is the solution of the Lyapunov equation

$$P(A+BK) + (A+BK)^{\mathrm{T}} P = -Q, \tag{10}$$

and is summarized in the following theorem.

Theorem 1. *There exist positive constants G and g such that the solutions of the closed-loop system (1), (2) satisfy $\Gamma(t) \le G e^{-gt} \Gamma(0)$ for all $t \ge 0$, where*

$$\Gamma(t) = |X(t)|^2 + \int_0^D u(x,t)^2 dx. \tag{11}$$

In the literature on delay systems the representation through the transport PDE state (5) is somewhat non-standard. The constructions provided in the transport PDE notation can also be expressed in the delay notation, such that the Lyapunov functional (9) is written as

$$V(t) = X(t)^T P X(t) + 2 \frac{|PB|^2}{\lambda_{\min}(Q)} \int_{t-D}^t (1+\theta+D-t) W(\theta)^2 d\theta, \tag{12}$$

and the backstepping transformation (3) is

$$W(\theta) = U(\theta) - K \left[\int_{t-D}^{\theta} e^{A(\theta-\sigma)} B U(\sigma) d\sigma + e^{A(\theta+D-t)} X(t) \right], \tag{13}$$

with $-D \le t-D \le \theta \le t$. We pursue the PDE notation for delay systems so we can seamlessly transition to PDE problems in the subsequent sections of the article.

The ability to construct a Lyapunov functional can be exploited in various ways, including deriving disturbance attenuation estimates when the system (1) is subject to an additive disturbance, proving robustness to a small actuator lag, and conducting an inverse optimal redesign of the predictor feedback. We consider these three problems in the current section. In subsequent sections, we present more substantial benefits of constructing a Lyapunov functional and a backstepping transformation. These benefits are the establishment of robustness to a small error in D, where the error is allowed to be either positive or negative, the design of adaptive controllers in the presence of a completely unknown and arbitrarily long D, the design of stabilizing predictor feedback for time varying delays, and the design of predictor feedback for some classes of nonlinear and PDE systems.

We now consider the system

$$\dot{X}(t) = AX(t) + BU(t-D) + B_1 d(t), \tag{14}$$

where $d(t)$ is an unmeasurable disturbance which is bounded but its bound is unknown, and the controller

$$U(t) = \frac{c}{s+c} \left\{ K \left[e^{AD} X(t) + \int_{t-D}^t e^{A(t-\theta)} B U(\theta) d\theta \right] \right\}, \tag{15}$$

where $c > 0$, and where we use the transfer function representation for compactness of notation.

The following result is established with the Lyapunov functional

$$V(t) = X(t)^{\mathrm{T}} P X(t) + 2 \frac{|PB|^2}{\lambda_{\min}(Q)} \int_0^D (1+x) w(x,t)^2 \, dx + \frac{1}{2} w(D,t)^2. \tag{16}$$

Theorem 2. *There exists a positive constant c^* such that for all $c > c^*$, the feedback system (14), (15) is L_∞-stable, that is, there exist positive constants β_1, β_2, γ_1 such that*

$$N(t) \le \beta_1 e^{-\beta_2 t} N(0) + \gamma_1 \sup_{\tau \in [0,t]} |d(\tau)|, \tag{17}$$

where

$$N(t) = \left(|X(t)|^2 + \int_{t-D}^t U(\theta)^2 d\theta + U(t)^2 \right)^{1/2}. \tag{18}$$

Furthermore, there exists a constant $c^{**} > c^*$ such that for all $c \ge c^{**}$ the feedback (15) minimizes the cost functional

$$J = \sup_{d \in \mathcal{D}} \lim_{t \to \infty} \left[2cV(t) + \int_0^t (Q(\tau) + \dot{U}(t)^2 - c\gamma_2 d(\tau)^2) d\tau \right], \tag{19}$$

for each

$$\gamma_2 \ge \gamma_2^{**} = 8 \frac{|PB|^2}{\lambda_{\min}(Q)}, \tag{20}$$

where $Q(t) \ge \mu N(t)^2$ for some $\mu(c, \gamma_2) > 0$, which is such that $\mu(c, \gamma_2) \to \infty$ as $c \to \infty$, and $\mathcal{D}$ is the set of linear scalar-valued functions of X.

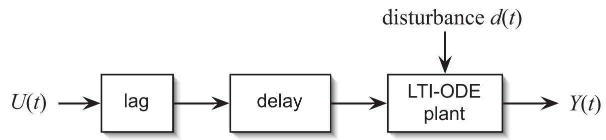

Fig. 1. An ODE with input delay and with an unmodeled input lag and additive disturbance. A suitable form of robustness holds with respect to both perturbations under predictor feedback (2), as stated in Theorem 2.

The following four special cases can be inferred from Theorem 2. First, the predictor feedback (2) is robust to the introduction of a lag $c/(s+c)$ for sufficiently high c. The lag can be either a part of the control law, as in (15), or an unmodeled part of the system dynamics, as shown in Fig. 1. Second, the system under predictor feedback (2), as well as under feedback (15) with sufficiently high c, has a finite L_∞ gain relative to an additive disturbance. Third, the feedback (15) is an inverse optimal stabilizer for sufficiently high but finite c, in the absence of the disturbance d. This property is not so easy to see. It is obtained by writing the feedback law in terms of $\dot{U}(t)$ as the control input, in which case the feedback law is of the form '$-L_gV$' (Sepulchre, Jankovic, & Kokotovic, 1997). Fourth, in the presence of the disturbance, the feedback (15) with sufficiently high c is an inverse optimal solution to a differential game problem (Krstic & Deng, 1998) with a positive definite penalty on the state and control, and a negative-definite penalty on the disturbance.

2.2. Multi-input systems with distributed delays

In this section we consider the system

$$\dot{X}(t) = AX(t) + \int_0^{D_1} B_1(\sigma)U_1(t-\sigma)d\sigma + \int_0^{D_2} B_2(\sigma)U_2(t-\sigma)d\sigma. \tag{21}$$

In this case a backstepping transformation as in (3) and (4) is not applicable since the system comprised of the finite-dimensional state of the plant $X(t)$ and the infinite-dimensional actuator states $U_1(t+x-D_1), x\in[0,D_1]$ and $U_2(t+z-D_2), z\in[0,D_2]$, is not in the strict-feedback form.

For system (21), under the controllability condition of the pair $(A,[B_{D_1}\quad B_{D_2}])$ with

$$B_{D_i} = \int_0^{D_i} e^{-A\sigma}B_i(\sigma)d\sigma, \quad i=1,2, \tag{22}$$

the controller developed in (Artstein, 1982; Kwon & Pearson, 1980; Manitius & Olbrot, 1979) which achieves asymptotic stabilization for any D_1, $D_2>0$, has the form

$$U(t) = \begin{bmatrix} U_1(t) \\ U_2(t) \end{bmatrix} = \begin{bmatrix} K_1 e^{AD_1} \\ K_2 e^{AD_2} \end{bmatrix} Z(t) \tag{23}$$

$$Z(t) = X(t) + \sum_{i=1}^{2}\int_0^{D_i}\int_{D_i-y}^{D_i} e^{A(D_i-y-\sigma)}B_i(\sigma)d\sigma \times U_i(t+y-D_i)dy, \quad i=1,2, \tag{24}$$

where the control gains K_1 and K_2 may be designed by an LQR/Riccati approach, pole-placement, or some other method that makes $A_{cl} = A + B_{D_1}K_1e^{AD_1} + B_{D_2}K_2e^{AD_2}$ Hurwitz.

To perform a closed-loop stability analysis, we denote the actuator state as

$$u_1(x,t) = U_1(t+x-D_1), \quad x\in[0,D_1] \tag{25}$$

$$u_2(z,t) = U_2(t+z-D_2), \quad z\in[0,D_2] \tag{26}$$

and introduce two infinite-dimensional transformations of the actuator states given by

$$\begin{aligned} w_1(x,t) = {} & u_1(x,t) - K_1e^{Ax}X(t) - \int_0^x\int_{D_1-y}^{D_1} K_1 \\ & \times e^{A(D_1+x-y-\sigma)}B_1(\sigma)d\sigma u_1(y,t)dy \\ & + \int_x^{D_1}\int_0^{D_1-y} K_1e^{A(D_1+x-y-\sigma)}B_1(\sigma)d\sigma u_1(y,t)dy \\ & - \int_0^{D_2}\int_{D_2-y}^{D_2} K_1e^{A(D_2+x-y-\sigma)}B_2(\sigma)d\sigma u_2(y,t)dy \end{aligned} \tag{27}$$

$$\begin{aligned} w_2(z,t) = {} & u_2(z,t) - K_2e^{Az}X(t) - \int_0^z\int_{D_2-y}^{D_2} K_2 \\ & \times e^{A(D_2+z-y-\sigma)}B_2(\sigma)d\sigma u_2(y,t)dy \\ & + \int_z^{D_2}\int_0^{D_2-y} K_2e^{A(D_2+z-y-\sigma)}B_2(\sigma)d\sigma u_2(y,t)dy \\ & - \int_0^{D_1}\int_{D_1-y}^{D_1} K_2e^{A(D_1+z-y-\sigma)}B_1(\sigma)d\sigma u_1(y,t)dy, \end{aligned} \tag{28}$$

together with the transformation of the finite-dimensional state $X(t)$ given in (24), to transform the system (21) into the *target system*

$$\dot{Z}(t) = A_{cl}Z(t) \tag{29}$$

$$\partial_t w_1(x,t) = \partial_x w_1(x,t) - q_1(x,D_2)K_2e^{AD_2}Z(t) \tag{30}$$

$$w_1(D_1,t) = 0 \tag{31}$$

$$\partial_t w_2(z,t) = \partial_z w_2(z,t) - q_2(z,D_1)K_1e^{AD_1}Z(t) \tag{32}$$

$$w_2(D_2,t) = 0. \tag{33}$$

The inverse transformations of (24) and (27)–(28) are

$$\begin{aligned} X(t) = {} & \left(I - \sum_{i=1}^{2}\int_0^{D_i}\int_{D_i-y}^{D_i} e^{A(D_i-y-\sigma)}B_i(\sigma)d\sigma K_i e^{(A+B_{ie}K_i)(y-D_i)}dy\right)Z(t) \\ & - \sum_{i=1}^{2}\int_0^{D_i}\int_{D_i-y}^{D_i} e^{A(D_i-y-\sigma)}B_i(\sigma)d\sigma(w_i(y,t) \\ & - \int_y^{D_i} K_ie^{(A+B_{ie}K_i)(y-r)}B_{ie}w_i(r,t)dr)dy \end{aligned} \tag{34}$$

$$\begin{aligned} u_1(x,t) = {} & w_1(x,t) + K_1e^{(A+B_{1e}K_1)(x-D_1)}e^{AD_1}Z(t) \\ & - \int_x^{D_1} K_1e^{(A+B_{1e}K_1)(x-y)}e^{(A+B_{1e}K_1)(x-y)}B_{1e}w_1(y,t)dy \end{aligned} \tag{35}$$

$$\begin{aligned} u_2(z,t) = {} & w_2(z,t) + K_2e^{(A+B_{2e}K_2)(z-D_2)}e^{AD_2}Z(t) - \int_z^{D_2} K_2 \\ & \times e^{(A+B_{2e}K_2)(x-y)}B_{2e}w_2(y,t)dy, \end{aligned} \tag{36}$$

where

$$B_{ie} = e^{AD_i}B_{D_i}, \quad i=1,2. \tag{37}$$

Using a Lyapunov functional

$$V(t) = Z(t)^TPZ(t) + \alpha\int_0^{D_1}(1+x)w_1^2(x,t)dx + \beta\int_0^{D_2}(1+z)w_2^2(z,t)dz,$$

where the positive parameters α, β are appropriately chosen and $P = P^T > 0$ is the solution of the following Lyapunov equation

$$A_{cl}^T P + PA_{cl} = -Q, \tag{39}$$

for some $Q = Q^T > 0$, we arrive at the following result

Theorem 3. *Consider the closed-loop systems consisting of the plant (21) and the controller (23). Let the pair $(A, [B_{D_1} \quad B_{D_2}])$ be completely controllable and choose K_1 and K_2 such that $A + B_{D_1} K_1 e^{AD_1} + B_{D_2} K_2 e^{AD_2}$ is Hurwitz. There exist positive constants μ and ρ, such that*

$$\Omega(t) \le \mu \Omega(0) e^{-\rho t} \tag{40}$$

$$\Omega(t) = |X(t)|^2 + \sum_{i=1}^{2} \int_0^{D_i} U_i^2(t-\theta) d\theta. \tag{41}$$

3. Delay-robustness, delay-adaptivity, and time-varying delays

In control systems with input delay, the length of the delay is the most significant uncertainty, affecting robustness to a small mismatch in the delay D when designing constant predictor feedback. It is also crucial in the design of delay-adaptive predictor feedback for a large uncertainty in the delay D.

3.1. Robustness to delay mismatch

We first discuss the problem of robustness to delay mismatch ΔD, as depicted in Fig. 2, and consider the feedback system

$$\dot{X}(t) = AX(t) + BU(t - D_0 - \Delta D), \tag{42}$$

$$U(t) = K\left[e^{AD_0} X(t) + \int_{t-D_0}^{t} e^{A(t-\theta)} BU(\theta) d\theta\right]. \tag{43}$$

The delay mismatch ΔD can be either positive or negative relative to the assumed actuator delay $D_0 > 0$. However, the actual delay must be nonnegative, $D_0 + \Delta D \ge 0$. For the study of robustness to a small ΔD, we use two different Lyapunov functionals, one for $\Delta D > 0$, which is the easier of the two cases, and another for $\Delta D < 0$, in which case we employ

$$V(t) = X(t)^{\mathrm{T}} PX(t) + \frac{a}{2} \int_0^{D_0 + \Delta D} (1 + x) w(x,t)^2 dx + \frac{1}{2} \int_{\Delta D}^{0} (D_0 + x) w(x,t)^2 dx \tag{44}$$

with a sufficiently large a.

Theorem 4. *There exists a positive constant δ such that for all $\Delta D \in (-\delta, \delta)$ there exist positive constants G and g such that the solutions of the closed-loop system (42), (43) satisfy $\Gamma(t) \le G e^{-gt} \Gamma(0)$ for all $t \ge 0$, where*

$$\Gamma(t) = |X(t)|^2 + \int_{t-\bar{D}}^{t} U(\theta)^2 d\theta \tag{45}$$

and where

$$\bar{D} = D_0 + \max\{0, \Delta D\}. \tag{46}$$

The significance of this robustness result can be assessed based on the intuition drawn from existing results. For example, the result (Teel, 1998) that finite-dimensional feedback laws for finite-dimensional plants are robust to small delays does not apply to our

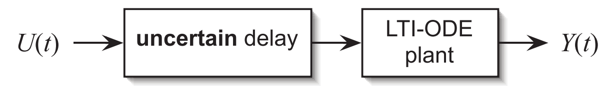

Fig. 2. An ODE with input delay which is known up to a small mismatch error ΔD, which can be either positive or negative. Stability is preserved under predictor feedback (43) for sufficiently small $|\Delta D|$ but arbitrarily large D, as stated in Theorem 4.

infinite-dimensional problem. The delay perturbation to predictor feedback incorporates the possibility of two different classes of perturbations, depending on whether ΔD is positive or negative, so existing results cannot be used.

The result of Theorem 4 may be surprising in light of negative result on delay-robustness for certain examples of hyperbolic PDEs with boundary control (Datko, 1988). Even though the input-delay problem also involves a hyperbolic PDE, such a negative result does not hold for predictor feedback because of a significant difference between first-order and second-order hyperbolic PDEs. The second-order hyperbolic PDEs in Datko's work have infinitely many eigenvalues on the imaginary axis, whereas this is not the case with an ODE with input delay. Even when the ODE is unstable, only a finite number of open-loop eigenvalues may be in the closed right-half plane.

3.2. Delay-adaptive control

Now we turn our attention from robustness to small delay mismatch to adaptivity for large delay uncertainty. Several results exist on adaptive control of systems with known input delays, including (Niculescu & Annaswamy, 2003; Ortega & Lozano, 1988). However, existing results deal with parametric uncertainties in the ODE plant, whereas the key challenge is uncertainty in the delay.

Let us consider the plant (1) but with a transport PDE representation of the input delay given as

$$\dot{X}(t) = AX(t) + Bu(0,t), \tag{47}$$

$$Du_t(x,t) = u_x(x,t), \tag{48}$$

$$u(1,t) = U(t). \tag{49}$$

We take the predictor feedback in the certainty equivalence form

$$U(t) = K\left[e^{A\hat{D}(t)} X(t) + \hat{D}(t) \int_0^1 e^{A\hat{D}(t)(1-y)} Bu(y,t) dy\right], \tag{50}$$

where the update law for the estimate $\hat{D}(t)$ is designed as

$$\dot{\hat{D}}(t) = \gamma \mathrm{Proj}_{[0,\bar{D}]}\{\tau(t)\}, \tag{51}$$

$$\tau(t) = -\frac{\int_0^1 (1+x) w(x,t) K}{\Gamma(t)} e^{A\hat{D}(t)x} dx (AX(t) + Bu(0,t)) \tag{52}$$

$$\Gamma(t) = 1 + X(t)^T PX(t) + b \int_0^1 (1+x) w(x,t)^2 dx \tag{53}$$

$$w(x,t) = u(x,t) - \hat{D}(t) \int_0^x K e^{A\hat{D}(t)(x-y)} Bu(y,t) dy - K e^{A\hat{D}(t)x} X(t), \tag{54}$$

with $b \ge (4\bar{D}|PB|^2)/(\lambda_{\min}(Q))$, where $\bar{D}$ is an a priori known upper bound on D. The standard projection operator projects $\hat{D}(t)$ into the interval $[0, \bar{D}]$. The structure of the adaptive control system is shown in Fig. 3. The choice of the update law (51) and (52) is motivated by a rather subtle Lyapunov analysis, resulting in a

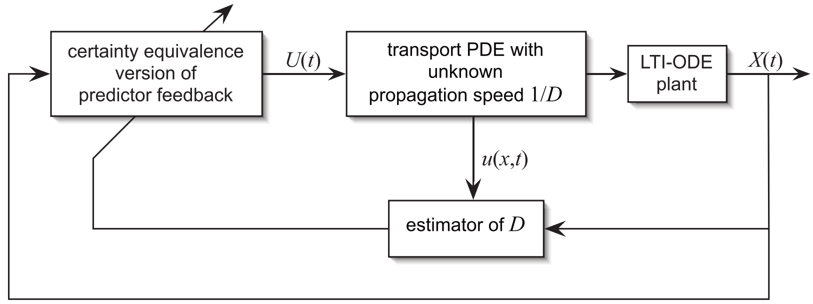

Fig. 3. Delay-adaptive predictor feedback for a true delay D varying in a broad range from 0 to a possibly large value $\bar{D}$. The certainty-equivalence controller (50) is combined with the update law (51)–(52). Global stability and regulation of the state and control are achieved, as specified in Theorem 5.

normalization of the update law, without the use of any filters or overparametrization.

Theorem 5. *Consider the closed-loop adaptive system (47)–(52). There exists $\gamma^* > 0$ such that for all $\gamma \in (0, \gamma^*)$ there exist positive constants R and ρ (independent of the initial conditions) such that for all initial conditions satisfying $(X_0, u_0, \hat{D}_0) \in \mathbb{R}^n \times L_2[0,1] \times [0, \bar{D}]$, the norm of the solutions obeys an exponential bound relative to the norm of initial conditions, namely*

$$\Upsilon(t) \le R\left(e^{\rho \Upsilon(0)} - 1\right), \quad \text{for all } t \ge 0, \tag{55}$$

where

$$\Upsilon(t) = |X(t)|^2 + \int_0^1 u(x,t)^2 dx + (D - \hat{D}(t))^2. \tag{56}$$

Furthermore

$$\lim_{t\to\infty} X(t) = 0, \quad \lim_{t\to\infty} U(t) = 0. \tag{57}$$

Example 3.1. We illustrate the delay-adaptive design for the unstable plant

$$X(s) = \frac{e^{-s}}{(s - 0.75)} U(s) \tag{58}$$

with the simulation results given in Fig. 4. The interval up to 1 s, in which the state $X(t)$ grows exponentially, is the result of input delay, which is 1 s. The parameter estimation is active until about 3 s. The control evolution is exponential (corresponding to a predominantly LTI system) after 3 s. The state decays exponentially after 4 s, namely, after the predominantly linear feedback has passed through the 1 s input delay. The adaptive controller is successful both with $\hat{D}(0) = 0$ and with $\hat{D}(0) = 2D$ (100% parameter error in both cases).

The controller (50)–(52) uses full state measurement of the transport PDE state. In the absence of such measurement, a slightly different design guarantees local stability, which is the strongest result achievable in that case due to a nonlinear parametrization of the operator e^{-Ds}.

3.3. Time-varying input delay

Before we close this section on uncertain delays, let us briefly turn our attention to the problem of known *time-varying* input delays (Fig. 5). We consider the system

$$\dot{X}(t) = AX(t) + BU(\phi(t)). \tag{59}$$

A predictor feedback for this system is

$$U(t) = K\left[e^{A(\phi^{-1}(t)-t)}X(t) + \int_{\phi(t)}^{t} e^{A(\phi^{-1}(t)-\phi^{-1}(\theta))}B\frac{U(\theta)}{\phi'(\phi^{-1}(\theta))}d\theta\right], \quad \text{for all} \quad t \ge 0. \tag{60}$$

with rather extensive effort, going through a transport PDE representation with $u(x, t) = U(\phi(t + x(\phi^{-1}(t) - t)))$ and the time-varying backstepping transformation

$$w(x,t) = u(x,t) - Ke^{Ax(\phi^{-1}(t)-t)}X(t) - K\int_0^x e^{A(x-y)(\phi^{-1}(t)-t)}Bu(y,t)(\phi^{-1}(t)-t)dy \tag{61}$$

into the target system

$$\dot{X}(t) = (A + BK)X(t) + Bw(0,t), \tag{62}$$

$$w_t(x,t) = \pi(x,t)w_x(x,t), \tag{63}$$

$$w(1,t) = 0, \tag{64}$$

where the variable speed of propagation of the transport equation w is given by

$$\pi(x,t) = \frac{1 + x(d(\phi^{-1}(t))/dt - 1)}{\phi^{-1}(t) - t}, \tag{65}$$

we obtain the following stabilization result.

Theorem 6. *Consider the closed-loop system (59), (60). Let the delay function $\delta(t) = t - \phi(t)$ be strictly positive and uniformly bounded from above. Let the delay rate function $\delta'(t)$ be strictly smaller than 1 and uniformly bounded from below. There exist positive constants G and g (the latter one being independent of ϕ) such that*

$$|X(t)|^2 + \int_{\phi(t)}^{t} U^2(\theta)d\theta \le Ge^{-gt}\left(|X_0|^2 + \int_{\phi(0)}^{0} U^2(\theta)d\theta\right), \quad \text{for all} \quad t \ge 0. \tag{66}$$

4. Predictor feedback for nonlinear systems

In robust nonlinear control several types of uncertainties are considered, including unmeasurable disturbances, uncertain static nonlinearities, unmodeled dynamics acting on the state, and unmodeled dynamics acting on the input, which have been the most significant challenge. Considerable success has been achieved with control of nonlinear systems with *state* delays (Germani,

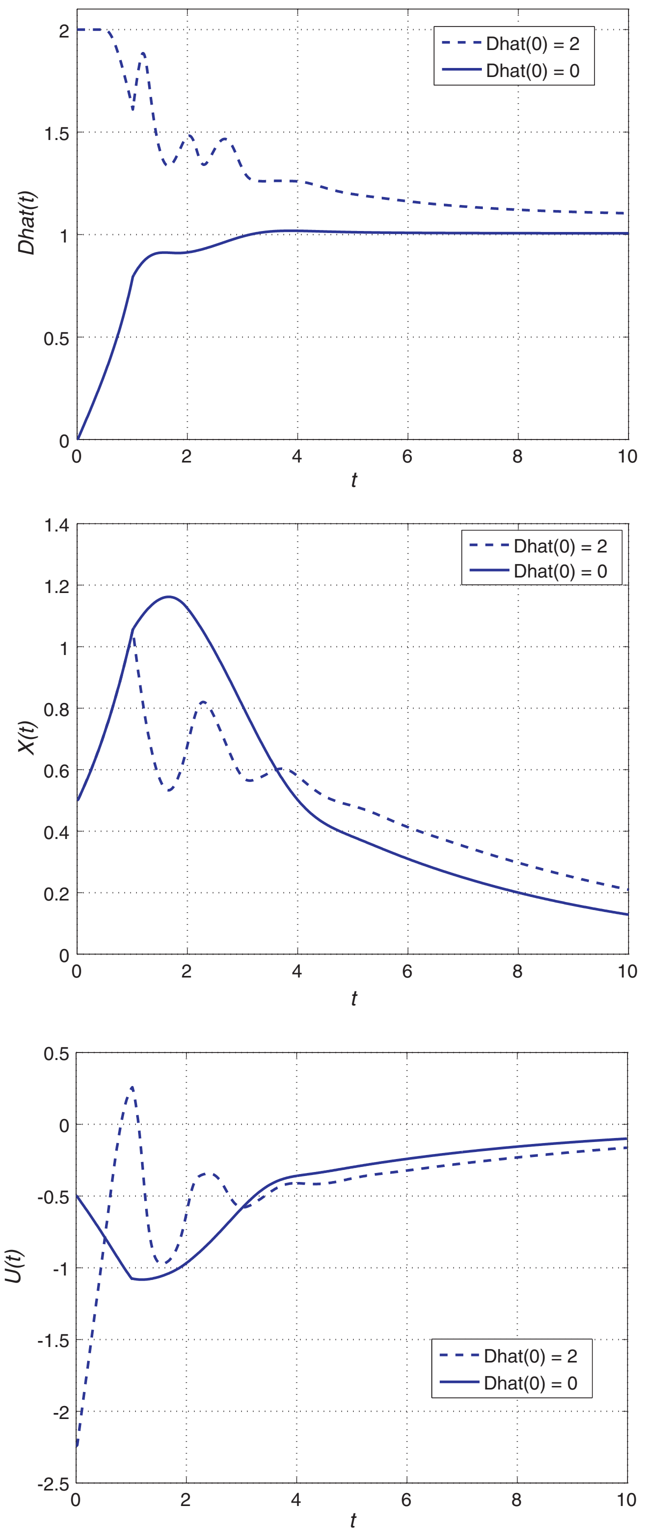

Fig. 4. Time responses of $\hat{D}(t)$, $X(t)$, and $U(t)$ under delay-adaptive predictor feedback for an unstable first-order plant. Stabilization is achieved both with $\hat{D}(0) = 0$ and with a $\hat{D}(0)$ that heavily overestimates the true D.

Manes, & Pepe, 2003; Jankovic, 2001; Karafyllis, 2006; Mazenc & Bilman, 2006) and one result was even developed for robustness to input delay of arbitrary length for feedforward systems (Mazenc, Mondie, & Francisco, 2004). However, systematic compensation of long delays at the *input* of nonlinear control systems, as depicted in Fig. 6, has never been considered.

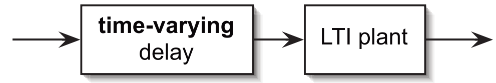

Fig. 5. Linear system $\dot{X}(t) = AX(t) + BU(\phi(t))$ with time-varying actuator delay $\delta(t) = t - \phi(t)$. The predictor feedback (60) with compensation of the time-varying delay achieves exponential stabilization according to Theorem 6.

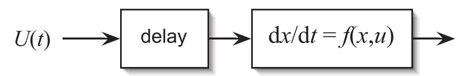

Fig. 6. Nonlinear control in the presence of arbitrarily long input delay. Global stabilization is achieved with the predictor feedback (68)–(70) if the plant is forward complete and globally asymptotically stabilizable in the absence of delay, as stated in Theorem 7.

An approach to compensate input delays in nonlinear control is through an extension of predictor feedback to nonlinear systems, which we present next. Consider the nonlinear system

$$\dot{X}(t) = f(X(t), U(t-D)), \quad f(0,0) = 0, \tag{67}$$

and assume that a feedback law $U = \kappa(X)$ with $\kappa(0) = 0$ is known which globally asymptotically stabilizes the system at the origin when $D = 0$. Denote the initial conditions as $Z_0 = Z(0)$ and $U_0(\theta) = U(\theta)$, $\theta \in [-D, 0]$. A predictor feedback is given by

$$U(t) = \kappa(P(t)), \tag{68}$$

where the predictor state is defined as

$$P(t) = \int_{t-D}^{t} f(P(\theta), U(\theta))d\theta + Z(t), \quad t \geq 0, \tag{69}$$

$$P(\theta) = \int_{-D}^{\theta} f(P(\sigma), U_0(\sigma))d\sigma + Z_0, \quad \theta \in [-D, 0]. \tag{70}$$

A key feature of the predictor $P(t)$ is that it is defined implicitly, through a nonlinear integral equation, rather than explicitly, through matrix exponentials and the variation of constants formula, as is the case when the plant is linear. The lack of an explicit formula for $P(t)$ is not an obstacle, since $P(t)$ is defined in terms of its past values.

The nonlinear predictor design is developed for systems that do not exhibit a finite escape time for any initial condition and any input signals that remain finite over finite time intervals (*forward complete* systems), which includes many mechanical and other systems, predictor feedback is developed which achieves global asymptotic stability, as long as the system without delay is globally asymptotically stabilizable. The predictror requires the solution of a nonlinear integral equation, or a nonlinear DDE, in real time.

Theorem 7. *Let $\dot{X} = f(X, U)$ be forward complete and $\dot{X} = f(X, \kappa(X))$ be globally asymptotically stable at $X = 0$. Consider the closed-loop system (67)–(70). There exists a function $\hat{\beta} \in \mathcal{KL}$ such that*

$$\Omega(t) \leq \hat{\beta}(\Omega(0), t) \tag{71}$$

$$\Omega(t) = |Z(t)| + \|U\|_{L_\infty[t-D,t]} \tag{72}$$

for all $(Z_0, U_0) \in \mathbb{R}^n \times L_\infty[-D, 0]$ and for all $t \geq 0$.

A significant class of nonlinear system exists for which $P(t)$ is explicitly computable. This is the class of *strict-feedforward* systems (Sepulchre et al., 1997).

Example 4.1. We illustrate the explicit computability of the predictor, and thus of the feedback law, for the third-order system

$$\dot{X}_1(t) = X_2(t) + X_3^2(t), \tag{73}$$

$$\dot{X}_2(t) = X_3(t) + X_3(t)U(t-D), \tag{74}$$

$$\dot{X}_3(t) = U(t-D), \tag{75}$$

which is not feedback linearizable, but is in the strict-feedforward class. The globally asymptotically stabilizing predictor feedback for this system is

$$U(t) = -P_1(t) - 3P_2(t) - 3P_3(t) - \frac{3}{8}P_2^2(t) + \frac{3}{4}P_3(t) \times \begin{pmatrix} -P_1(t) - 2P_2(t) + \frac{1}{2}P_3(t) + \frac{P_2(t)P_3(t)}{2} \\ + \frac{5}{8}P_3^2(t) - \frac{1}{4}P_3^3(t) - \frac{3}{8}\left(P_2(t) - \frac{P_3^2(t)}{2}\right)^2 \end{pmatrix}, \tag{76}$$

where the predictor of $(X_1(t), X_2(t), X_3(t))$ is given explicitly by

$$P_1(t) = X_1(t) + DX_2(t) + \frac{1}{2}D^2X_3(t) + DX_3^2(t) + 3X_3(t)\int_{t-D}^{t}(t-\theta)U(\theta)d\theta + \frac{1}{2}\int_{t-D}^{t}(t-\theta)^2U(\theta)d\theta + \frac{3}{2}\int_{t-D}^{t}\left(\int_{t-D}^{\theta}U(\sigma)d\sigma\right)^2 d\theta, \tag{77}$$

$$P_2(t) = X_2(t) + DX_3(t) + X_3(t)\int_{t-D}^{t}U(\theta)d\theta + \int_{t-D}^{t}(t-\theta)U(\theta)d\theta + \frac{1}{2}\left(\int_{t-D}^{t}U(\theta)d\theta\right)^2, \tag{78}$$

$$P_3(t) = X_3(t) + \int_{t-D}^{t}U(\theta)d\theta. \tag{79}$$

Note that the nonlinear infinite-dimensional feedback operator employs a finite Volterra series in $U(\theta)$.

5. Delay-PDE cascades

When a plant with an input delay is a PDE, such as in Fig. 7, special challenges arise in the design of predictor feedback, particularly if the PDE is actuated through boundary control, which makes the B operator unbounded. In (Krstic, 2009a) we consider two benchmark delay-PDE cascades, one where the plant is a parabolic PDE and the other where the plant is a second-order hyperbolic PDE. We review here the parabolic case, where the plant is an unstable reaction-diffusion equation with an arbitrarily large number of unstable eigenvalues in open loop.

Consider the PDE system

$$u_t(x,t) = u_{xx}(x,t) + \lambda u(x,t), \tag{80}$$

$$u(0,t) = 0, \tag{81}$$

$$u(1,t) = U(t-D), \tag{82}$$

where λ is an arbitrary constant. We derive a stabilizing feedback law in the explicit form

$$U(t) = 2\sum_{n=1}^{\infty}\int_0^1 \sin(\pi n\xi)\lambda\xi\frac{I_1\left(\sqrt{\lambda\left(1-\xi^2\right)}\right)}{\sqrt{\lambda\left(1-\xi^2\right)}}d\xi \times \begin{pmatrix} -e^{(\lambda-\pi^2n^2)D}\int_0^1 \sin(\pi n y)u(y,t)dy \\ +\pi n(-1)^n\int_{t-D}^{t} e^{(\lambda-\pi^2n^2)(t-\theta)} \times U(\theta)d\theta \end{pmatrix}, \tag{83}$$

where $I_1(\cdot)$ is a Bessel function.

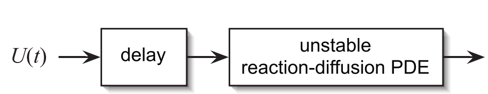

Fig. 7. Control of an unstable parabolic PDE with input delay, that is, of a boundary controlled cascade of a transport PDE and a reaction-diffusion PDE. Explicit gains are derived for the predictor feedback (83). As stated in Theorem 8, stability is achieved in a somewhat non-standard Sobolev norm, rather than in the basic L_2 norm of the state of the PDE cascade.

Theorem 8. *Consider the closed-loop system defined in (80)–(83). There exists a positive continuous function $\varrho : \mathbb{R}^2 \to \mathbb{R}_+$ such that, for all initial conditions $(u_0, U_0) \in L_2[0,1] \times H_1[0,D]$, and for all $c > 0$, the solutions are bounded as follows*

$$\Upsilon(t) \le \varrho(D,\lambda)e^{cD}\Upsilon(0)e^{-\min\{2,c\}t}, \quad \textit{for all} \quad t \ge 0, \tag{84}$$

where

$$\Upsilon(t) = \int_0^1 u^2(x,t)dx + \int_{t-D}^{t}\left(U^2(\theta) + \dot{U}^2(\theta)\right)d\theta. \tag{85}$$

Two features of this result are of significance and they arise in any application of predictor feedback to PDEs with boundary control. First, the feedback law (83) is derived explicitly. The explicit determination of the control gains is made possible by first deriving the control gain for $D = 0$ explicitly, which was achieved in (Smyshlyaev & Krstic, 2004), and then by solving the undriven version of the PDE system (80)–(82) with an initial condition given by the control gain for $D = 0$. In more specific terms, we solve the PDE systems

$$k_{xx}(x,y) = k_{yy}(x,y) + \lambda k(x,y), \quad 0 \le y \le x \le 1, \tag{86}$$

$$k(x,0) = 0, \tag{87}$$

$$k(x,x) = -\frac{\lambda}{2}x, \tag{88}$$

and

$$\gamma_x(x,y) = \gamma_{yy}(x,y) + \lambda\gamma(x,y), \quad (x,y) \in [1, 1+D] \times (0,1), \tag{89}$$

$$\gamma(x,0) = 0, \tag{90}$$

$$\gamma(x,1) = 0, \tag{91}$$

$$\gamma(1,y) = k(1,y). \tag{92}$$

Note that the k-system is hyperbolic and defined on a triangular domain, whereas the γ-system is parabolic and defined on a rectangular semi-infinite domain, as well as that the solution to the k-system acts as an initial condition to the γ-system, as given by (92). The process of explicitly solving for $\gamma(x, y)$ is the PDE equivalent of analytically finding the vector Ke^{AD} in (2).

The second feature is that, when dealing with boundary control of a PDE with input delay, we are facing the problem of control of two PDEs from different classes, such as a parabolic PDE and a first-order hyperbolic PDE in the case covered here, where the PDEs are interconnected through a boundary. While for each one of the two PDEs individually a natural system norm may be the standard L_2 norm, for the interconnected system this may not be the case and a higher order norm may have to be used for one of the subsystems, as is the case in (85).

6. ODEs controlled through distributed diffusion with counter-convection or through wave PDEs

For single-input systems explicit feedback laws are constructed in (Krstic, 2009b, 2009c; Susto & Krstic, 2010), for input dynamics governed by diffusion, diffusion with counter-convection and string PDEs respectively, based on the PDE backstepping. Here we consider a more complex set of problems involving multi-input ODE systems in which the convection, diffusion, counter-convection or wave propagation speed coefficients of the inputs's dynamics are different in each individual input channel.

As also mentioned in Section 2.2, the PDE backstepping approach alone does not suffice in the case of distributed input dynamics (where the ODE's right hand side incorporates an integral of the actuator state). Backstepping also does not suffice in the case of multi-input ODEs with PDE actuator dynamics in which the convection, diffusion, counter-convection or wave propagation speed coefficients are different in each individual input channel. We combine here backstepping and forwarding to introduce invertible transformations not only of the actuator states but also of the state of the plant.

6.1. Diffusion with counter-convection

We consider the system

$$\dot{X}(t) = AX(t) + \int_0^{D_1} B_1(y)u_1(y,t)dy + \int_0^{D_2} B_2(y)u_2(y,t)dy \tag{93}$$

$$\partial_t u_1(x,t) = \partial_{xx}u_1(x,t) - b_1\partial_x u_1(x,t) \tag{94}$$

$$\partial_x u_1(0,t) = 0 \tag{95}$$

$$u_1(D_1,t) = U_1(t) \tag{96}$$

$$\partial_t u_2(z,t) = \partial_{zz}u_2(z,t) - b_2\partial_z u_2(z,t) \tag{97}$$

$$\partial_z u_2(0,t) = 0 \tag{98}$$

$$u_2(D_2,t) = U_2(t), \tag{99}$$

where $x \in [0, D_1]$, $z \in [0, D_2]$ and $b_1, b_2 > 0$. We refer to the b_1 and b_2 terms in (94) and (97), respectively, as counter-convection because the terms act in a manner opposite to the usual convection terms in the standard transport PDE. While convection promotes the motion of the control signal away ('downstream') from the boundary condition at which the input is applied, these counter-convection terms actually promote a backward ('up-stream') motion of the control signal. The only reason why the control signals $U_1(t)$ and $U_2(t)$ can actually propagate and reach the ODE in (93) is the presence of the diffusion terms in (94) and (97). While convection (and counter-convection) has a fixed propagation speed, diffusion is not subject to that limitation, so the effect of diffusion prevails and the control signals can reach the ODE, to stabilize it. However, the interplay between the diffusion and counter-convection is not only about the propagation direction and speed, but it is more complex. Convection is a simple transport process, which does not change the waveform of the input signal. Diffusion is more complex and its effect is of low-pass character, but the effect gets increasingly severe for signals of higher frequency because diffusion induces infinitely many eigenvalues extending all the way to infinity on the negative real axis. Hence, a controller whose task is to stabilize the ODE (93) faces two challenges associated with the actuator dynamics—these dynamics are of high relative degree due to diffusion and are unstable due to counter-convection.

For notational simplicity we consider a two-input case. The same control design and analysis can be carried out for an arbitrary number of inputs. We define the transformations

$$Z(t) = X(t) + \sum_{i=1}^{2}\int_0^{D_i} g_i(y)u_i(y,t)dy \tag{100}$$

$$w_1(x,t) = u_1(x,t) - \gamma_1(x)\left(X(t) + \sum_{i=1}^{2}\int_0^{D_i} g_i(y)u_i(y,t)dy\right) + \frac{b_1}{2}\int_0^x e^{b_1(x-y)}u_1(y,t)dy \tag{101}$$

$$w_2(z,t) = u_2(z,t) - \gamma_2(z)\left(X(t) + \sum_{i=1}^{2}\int_0^{D_i} g_i(y)u_i(y,t)dy\right) + \frac{b_2}{2}\int_0^z e^{b_2(z-y)}u_2(y,t)dy, \tag{102}$$

where

$$g_i(x) = [I \quad 0]\left(e^{A_i x}\begin{bmatrix} I \\ -b_i I\end{bmatrix} - \int_0^x e^{A_i(x-y)}\begin{bmatrix} 0 \\ I\end{bmatrix}\frac{b_i}{2}e^{b_i(D_i-y)}dyR_i\right)$$
$$F_i - [I \quad 0]\begin{pmatrix} \int_0^x e^{A_i(x-y)}\begin{bmatrix} 0 \\ I\end{bmatrix}B_i(y)dy \\ + \int_0^x e^{A_i(x-y)}\begin{bmatrix} 0 \\ I\end{bmatrix}\frac{b_i}{2}e^{b_i(D_i-y)}dyG_i\end{pmatrix} \tag{103}$$

$$\gamma_i(w) = K_i\Gamma_i\begin{bmatrix} I & \frac{b_i}{2}I\end{bmatrix}e^{\begin{bmatrix} 0 & A_{cl} \\ I & b_i I\end{bmatrix}w}\begin{bmatrix} I \\ 0\end{bmatrix} \tag{104}$$

$$A_i = \begin{bmatrix} 0 & I \\ A & -b_i I\end{bmatrix} \tag{105}$$

$$M_i = I + \int_0^{D_i}[0 \quad I]e^{A_i(D_i-y)}\begin{bmatrix} 0 \\ I\end{bmatrix}\frac{b_i}{2}e^{b_i(D_i-y)}dy \tag{106}$$

$$R_i = M_i^{-1}[0 \quad I]e^{A_iD_i}\begin{bmatrix} I \\ -b_i I\end{bmatrix} \tag{107}$$

$$G_i = -M_i^{-1}\int_0^{D_i}[0 \quad I]e^{A_i(D_i-y)}\begin{bmatrix} 0 \\ I\end{bmatrix}B_i(y)dy \tag{108}$$

$$E_i = [I \quad 0]e^{A_iD_i}\begin{bmatrix} I \\ -b_iI \end{bmatrix} - \int_0^{D_i}[I \quad 0]e^{A_i(D_i-y)}\begin{bmatrix} 0 \\ I \end{bmatrix}\frac{b_i}{2}e^{b_i(D_i-y)}dyR_i \tag{109}$$

$$F_i = E_i^{-1}\left(\begin{array}{l}\int_0^{D_i}[I \quad 0]e^{A_i(D_i-y)}\begin{bmatrix} 0 \\ I \end{bmatrix}B_i(y)dy + \int_0^{D_i}[I \quad 0] \\ \times e^{A_i(D_i-y)}\begin{bmatrix} 0 \\ I \end{bmatrix}\frac{b_i}{2}e^{b_i(D_i-y)}dyG_i\end{array}\right) \tag{110}$$

$$\Gamma_i = \Lambda_i^{-1} \tag{111}$$

$$\Lambda_i = \begin{bmatrix} I & \frac{b_i}{2}I \end{bmatrix}e^{\begin{bmatrix} 0 & A_{cl} \\ I & b_iI \end{bmatrix}D_i}\begin{bmatrix} I \\ 0 \end{bmatrix}, \quad i = 1, 2. \tag{112}$$

Transformations (100)–(102) convert system (93)–(99) into

$$\dot{Z}(t) = A_{cl}Z(t) \tag{113}$$

$$\partial_t w_1(x,t) = \partial_{xx}w_1(x,t) - b_1\partial_x w_1(x,t) \tag{114}$$

$$\partial_x w_1(0,t) = \frac{b_1}{2}w_1(0,t) \tag{115}$$

$$w_1(D_1,t) = 0 \tag{116}$$

$$\partial_t w_2(z,t) = \partial_{zz}w_2(z,t) - b_2\partial_z w_2(z,t) \tag{117}$$

$$\partial_z w_2(0,t) = \frac{b_2}{2}w_2(0,t) \tag{118}$$

$$w_2(D_2,t) = 0, \tag{119}$$

where

$$A_{cl} = A - g_1'(D_1)K_1 - g_2'(D_2)K_2, \tag{120}$$

and K_1, K_2 are chosen such that A_{cl} is Hurwitz. The transformations (100)–(102) completely decouple the finite-dimensional state of the plant $X(t)$ and the infinite-dimensional actuator states $u_1(x, t)$ and $u_2(z, t)$. The *target system* (113)–(119) is comprised of the exponentially stable Z-system and two exponentially stable PDE w_1 and w_2-systems which incorporate diffusion with counter-convection. The reason why these w-system are stabile in spite of the presence of counter-convection is that they have stabilizing boundary conditions (115) and (118).

The inverse transformations of (100)–(102) are given by

$$X(t) = \left(I - \sum_{i=1}^{2}\int_0^{D_i} g_i(y)\delta_i(y)\right)Z(t) - \sum_{i=1}^{2}\int_0^{D_i} g_i(y)\left(w_i(y,t) - \frac{b_i}{2}\int_0^{y} e^{(b_i/2)(y-r)}w_i(r,t)dr\right)dy \tag{121}$$

$$u_1(x,t) = w_1(x,t) + \delta_1(x)Z(t) - \frac{b_1}{2}\int_0^{x} e^{(b_1/2)(x-y)}w_1(y,t)dy \tag{122}$$

$$u_2(z,t) = w_2(z,t) + \delta_2(z)Z(t) - \frac{b_2}{2}\int_0^{z} e^{(b_2/2)(z-y)}w_2(y,t)dy, \tag{123}$$

where

$$\delta_i(w) = K_i\Gamma_i[I \quad 0]e^{\begin{bmatrix} 0 & A_{cl} \\ I & b_iI \end{bmatrix}w}\begin{bmatrix} I \\ 0 \end{bmatrix}, \quad i = 1, 2. \tag{124}$$

Using the Lyapunov functional

$$V(t) = Z(t)^{\mathrm{T}}PZ(t) + \frac{1}{2}\int_0^{D_1} w_1(x,t)^2dx + \frac{1}{2}\int_0^{D_2} w_2(z,t)^2dz, \tag{125}$$

where $P = P^T > 0$ and $Q = Q^T > 0$ satisfy

$$A_{cl}^TP + PA_{cl} = -Q,$$

the following result can be proved.

Theorem 9. *Consider the closed-loop system consisting of the plant (93)–(99) and the control laws*

$$U_1(t) = K_1Z(t) - \frac{b_1}{2}\int_0^{D_1} e^{b_1(D_1-y)}u_1(y,t)dy \tag{126}$$

$$U_2(t) = K_2Z(t) - \frac{b_2}{2}\int_0^{D_2} e^{b_2(D_2-y)}u_2(y,t)dy. \tag{127}$$

Let the pair $(A, [g_1'(D_1)g_2'(D_2)])$ *be completely controllable and let the matrices* M_i, R_i, $i = 1, 2$ *be invertible. Choose* K_1 *and* K_2 *such that the matrix* A_{cl} *is Hurwitz and such that* Λ_i, $i = 1, 2$ *are invertible. Then the closed-loop system is exponentially stable in the sense that there exist positive constants* η *and* ν *such that*

$$\Omega(t) \le \eta\Omega(0)e^{-\nu t} \tag{128}$$

$$\Omega(t) = |X(t)|^2 + \int_0^{D_1} u_1(x,t)^2dx + \int_0^{D_2} u_2(z,t)^2dz. \tag{129}$$

An interesting special case of the system (93)–(99) is when $b_i = 0$, $i = 1, 2$, that is, when the inputs to the plant satisfy diffusion equations. In this case Theorem 9 applies by setting $b_i = 0$, $i = 1, 2$ in Eqs. (126)–(127). It is important here to observe that the direct transformations (101)–(102) are significantly simplified when we set $b_i = 0$, $i = 1, 2$. Moreover, the inverse transformations (122)–(123) are trivially satisfied with $\delta_i(\cdot) = \gamma_i(\cdot)$, $i = 1, 2$. One can see this by looking at the expressions (104) and (124) when $b_i = 0$, $i = 1, 2$, or by observing that (101) and (102) when $b_i = 0$, $i = 1, 2$, can be written as

$$w_1(x,t) = u_1(x,t) - \gamma_1(x)Z(t) \tag{130}$$

$$w_2(z,t) = u_2(z,t) - \gamma_2(z)Z(t). \tag{131}$$

6.2. Wave dynamics

In this section we consider the following system

$$\dot{X}(t) = AX(t) + \sum_{i=1}^{2}\left(\int_0^{D_i} B_i(y)u_i(y,t)dy + \int_0^{D_i} B_{it}(y)\partial_t u_i(y,t)dy\right) \tag{132}$$

$$\partial_{tt}u_1(x,t) = \partial_{xx}u_1(x,t) \tag{133}$$

$$\partial_x u_1(0,t) = 0 \tag{134}$$

$$\partial_x u_1(D_1,t) = U_1(t) \tag{135}$$

$$\partial_{tt} u_2(z,t) = \partial_{zz} u_2(z,t) \tag{136}$$

$$\partial_z u_2(0,t) = 0 \tag{137}$$

$$\partial_z u_2(D_2,t) = U_2(t), \tag{138}$$

where $x \in [0, D_1]$, $z \in [0, D_2]$, $D_1, D_2 > 0$. Define the transformations

$$Z(t) = X(t) + \sum_{i=1}^{2} \times \left(\int_0^{D_i} (Ag_i(y) - B_{it}(y) + g_i(D_i)c_{0i}c_{1i})u_i(y,t)dy + \int_0^{D_i} g_i(y)\partial_t u_i(y,t)dy + c_{1i}g_i(D_i)u_i(D_i,t) \right) \tag{139}$$

$$\Delta_i = [\,0 \quad I\,]\left(I + \int_0^{D_i} \mathrm{e}^{\begin{bmatrix} 0 & I \\ A^2 & 0 \end{bmatrix}(D_i - r)} \begin{bmatrix} 0 \\ I \end{bmatrix} dy A c_{1i} c_{0i} G_i^{-1} [\,I \quad 0\,] \right) + (c_{0i}I + c_{1i}A)G_i^{-1}[\,I \quad 0\,] \tag{147}$$

$$E_i = \Delta_i \mathrm{e}^{\begin{bmatrix} 0 & I \\ A^2 & 0 \end{bmatrix} D_i} \begin{bmatrix} I \\ 0 \end{bmatrix} \tag{148}$$

$$\gamma_1(x) = C_1 [\,I \quad c_{01}I\,] \mathrm{e}^{\begin{bmatrix} 0 & A_{cl}^2 \\ I & 0 \end{bmatrix} x} \begin{bmatrix} I \\ 0 \end{bmatrix} \tag{149}$$

$$R_i = [\,I \quad c_{0i}I\,] \mathrm{e}^{\begin{bmatrix} 0 & A_{cl}^2 \\ I & 0 \end{bmatrix} D_i} \begin{bmatrix} c_{1i}A_{cl} \\ I \end{bmatrix} \tag{150}$$

$$A_{cl} = A + g_1(D_1)K_1 + g_2(D_2)K_2 \tag{151}$$

$$w_1(x,t) = u_1(x,t) - \gamma_1(x)\left(X(t) + \sum_{i=1}^{2} \left(\int_0^{D_i} (Ag_i(y) - B_{it}(y) + g_i(D_i)c_{0i}c_{1i})u_i(y,t)dy + c_{1i}g_i(D_i)u_i(D_i,t) \right) + c_{1i}g_i(D_i)u_i(D_i,t) \right) + c_{01}\int_0^x u_1(y,t)dy \tag{140}$$

$$w_2(z,t) = u_2(z,t) - \gamma_2(z)\left(X(t) + \sum_{i=1}^{2} \left(\int_0^{D_i} (Ag_i(y) - B_{it}(y) + g_i(D_i)c_{0i}c_{1i})u_i(y,t)dy + \int_0^{D_i} g_i(y)\partial_t u_i(y,t)dy \right) + c_{1i}g_i(D_i)u_i(D_i,t) \right) + c_{02}\int_0^z u_2(y,t)dy, \tag{141}$$

$$\begin{aligned} g_i(y) = {} & [\,I \quad 0\,]\mathrm{e}^{\begin{bmatrix} 0 & I \\ A^2 & 0 \end{bmatrix} y} \left(I + \int_0^y \mathrm{e}^{-\begin{bmatrix} 0 & I \\ A^2 & 0 \end{bmatrix} r} dr \begin{bmatrix} 0 \\ I \end{bmatrix} A c_{1i} c_{0i} G_i^{-1} [\,I \quad 0\,] \mathrm{e}^{\begin{bmatrix} 0 & I \\ A^2 & 0 \end{bmatrix} D_i} \right) \begin{bmatrix} I \\ 0 \end{bmatrix} g_i(0) - [\,I \quad 0\,] \\ & \times \int_0^y \mathrm{e}^{\begin{bmatrix} 0 & I \\ A^2 & 0 \end{bmatrix}(y-r)} \begin{bmatrix} 0 \\ I \end{bmatrix} (B_i(r) + AB_{it}(r))dr - [\,I \quad 0\,] \int_0^y \mathrm{e}^{\begin{bmatrix} 0 & I \\ A^2 & 0 \end{bmatrix}(y-r)} dr \begin{bmatrix} 0 \\ I \end{bmatrix} A c_{1i} c_{0i} G_i^{-1} [\,I \quad 0\,] \int_0^{D_i} \mathrm{e}^{\begin{bmatrix} 0 & I \\ A^2 & 0 \end{bmatrix}(D_i - y)} \begin{bmatrix} 0 \\ I \end{bmatrix} \\ & \times (B_i(y) + AB_{it}(y))dy \end{aligned} \tag{142}$$

where

$$\gamma_i(w) = C_i [\,I \quad c_{0i}I\,] \mathrm{e}^{\begin{bmatrix} 0 & A_{cl}^2 \\ I & 0 \end{bmatrix} w} \begin{bmatrix} I \\ 0 \end{bmatrix} \tag{143}$$

$$\delta_i(w) = [\,C_i \quad 0\,] \mathrm{e}^{\begin{bmatrix} 0 & A_{cl}^2 \\ I & 0 \end{bmatrix} w} \begin{bmatrix} I \\ 0 \end{bmatrix} \tag{144}$$

$$G_i = I - [\,I \quad 0\,] \int_0^{D_i} \mathrm{e}^{\begin{bmatrix} 0 & I \\ A^2 & 0 \end{bmatrix}(D_i - r)} \begin{bmatrix} 0 \\ I \end{bmatrix} dr A c_{1i} c_{0i} \tag{145}$$

$$g_i(0) = E_i^{-1}\Delta_i \int_0^{D_i} \mathrm{e}^{\begin{bmatrix} 0 & I \\ A^2 & 0 \end{bmatrix}(D_i - r)} \begin{bmatrix} 0 \\ I \end{bmatrix} (B_i(r) + AB_{it}(r))dr \tag{146}$$

$$C_i = K_i R_i^{-1}, \quad i = 1, 2, \tag{152}$$

K_1 and K_2 are chosen such that A_{cl} is Hurwitz and c_{0i}, c_{1i}, $i = 1, 2$ are positive constants. Using (139)–(141) we transform system (132)–(138) into the *target system*

$$\dot{Z}(t) = A_{cl}Z(t) \tag{153}$$

$$\partial_{tt} w_1(x,t) = \partial_{xx} w_1(x,t) \tag{154}$$

$$\partial_x w_1(0,t) = c_{01} w_1(0,t) \tag{155}$$

$$\partial_x w_1(D_1,t) = -c_{11}\partial_t w_1(D_1,t) \tag{156}$$

$$\partial_{tt} w_2(z,t) = \partial_{zz} w_2(z,t) \tag{157}$$

$$\partial_z w_2(0,t) = c_{02} w_2(0,t) \tag{158}$$

$$\partial_z w_2(D_2,t) = -c_{12}\partial_t w_2(D_2,t). \quad (159)$$

This target system is exponentially stable because of it consists of decoupled Z-dynamics, which are exponentially stable, and w-dynamics, which are given by two wave equations with boundary damping, which are also exponentially stable.

The inverse transformations of (139)–(141) are

$$X(t) = \left(I - \sum_{i=1}^{2}\left(\int_0^{D_i}(Ag_i(y) - B_{it}(y) + g_i(D_i)c_{0i}c_{1i})\delta_i(y)dy - \int_0^{D_i} g_i(y)\delta_i(y)A_{cl}dy - c_{1i}g_i(D_i)\delta_i(D_i)\right)\right)Z(t)$$
$$- \sum_{i=1}^{2}\left(\begin{array}{c}\int_0^{D_i}(Ag_i(y) - B_{it}(y) + g_i(D_i)c_{0i}c_{1i})\left(w_i(y,t) - c_{0i}\int_0^{y} e^{-c_{0i}(y-r)}w_i(r,t)dr\right)dy - \int_0^{D_i} g_i(y)\\ \left(\partial_t w_i(y,t) - c_{0i}\int_0^{y} e^{-c_{0i}(y-r)}\partial_t w_i(r,t)dr\right)dy - c_{1i}g_i(D_i)\left(w_i(D_i,t) - c_{0i}\int_0^{D_i} e^{-c_{0i}(D_i-y)}w_i(y,t)dy\right)\end{array}\right) \quad (160)$$

$$u_1(x,t) = w_1(x,t) + \delta_1(x)Z(t) - c_{01}\int_0^{x} e^{-c_{01}(x-y)}w_1(y,t)dy \quad (161)$$

$$u_2(z,t) = w_2(z,t) + \delta_2(z)Z(t) - c_{02}\int_0^{z} e^{-c_{02}(z-y)}w_2(y,t)dy. \quad (162)$$

Using the Lyapunov functional

$$V(t) = Z(t)^T PZ(t) + E(t), \quad (163)$$

where $P = P^T > 0$ and $Q = Q^T > 0$ satisfy

$$A_{cl}^T P + PA_{cl} = -Q, \quad (164)$$

and

$$E(t) = \sum_{i=1}^{2}\left(\begin{array}{c}\frac{1}{2}\left(c_{0i}w_i(0,t)^2 + \|\partial_y w_i(t)\|^2 + \|\partial_t w_i(t)\|^2\right)\\ +\epsilon_i\int_0^{D_i}(1+y)\times\partial_y w_i(y,t)\partial_t w_i(y,t)dy\end{array}\right), \quad (165)$$

where ϵ_i, $i = 1, 2$ are sufficiently small positive constants, we arrive at the following theorem

Theorem 10. *Consider the closed-loop system consisting of the plant (132)–(138) and the control laws*

$$U_1(t) = K_1 Z(t) - c_{01}\left(c_{11}\int_0^{D_1}\partial_t u_1(y,t)dy + u_1(D_1,t)\right) - c_{11}\partial_t u_1(D_1,t) \quad (166)$$

$$U_2(t) = K_2 Z(t) - c_{02}\left(c_{12}\int_0^{D_2}\partial_t u_2(y,t)dy + u_2(D_2,t)\right) - c_{12}\partial_t u_2(D_2,t). \quad (167)$$

Let the pair $(A, [g_1(D_1)g_2(D_2)])$ be completely controllable and choose the positive constants c_{0i}, c_{1i}, $i = 1, 2$ such that the matrices G_i, E_i, $i = 1, 2$ are invertible. Furthermore, choose K_1, K_2 such that A_{cl} is Hurwitz, and such that the matrices R_i, $i = 1, 2$ are invertible. Then the closed-loop system is exponentially stable in the sense that there exist positive constants κ and λ such that

$$\Omega(t) \le \kappa\Omega(0)e^{-\lambda t} \quad (168)$$

$$\Omega(t) = |X(t)|^2 + \sum_{i=1}^{2}\left(\int_0^{D_i}\partial_y u_i(y,t)^2 dy + \int_0^{D_i}\partial_t u_i(y,t)^2 dy + u_i(0,t)^2\right). \quad (169)$$

7. Conclusions

The PDE backstepping approach is a powerful tool in advancing the design techniques for systems with input and output delays.

Two techniques presented in this article are of interest to researchers in delay systems. The first technique is the construction of backstepping and forwarding transformations that allow the designer to deal with delays and PDE dynamics at the input, as well as in the main line of applying control action, such as in the chain of integrators for systems in triangular forms,

The second technique is the construction of Lyapunov functionals and explicit stability estimates, with the help of direct and inverse backstepping and forwarding transformations.

A wealth of future opportunities exist for research in this area

- Extending the explicit predictor feedback design to various classes of systems with simultaneous input and state delays.
- Developing predictor feedback for systems with state-dependent input delays, which are related to, but not a sub-class, of the case of time-varying delays.
- Developing design tools for nonlinear systems with input dynamics governed by heat or wave PDEs.
- Developing feedback laws for more general PDE-PDE cascades, such as wave-heat (or structure-fluid) and other interconnections.

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