IV. Conclusion

We revisit the singular control problem and obtain new numerical and theoretical results. The advantage of the obtained numerical algorithm is that it is robust in respect to the numerical inaccuracy, which is due to a numerical rounding. Indeed, if some incorrect FGEs are extracted due to the numerical inaccuracy instead of the correct IGEs, then we reject them since they do not appear in symmetric pairs. A numerical algorithm for the orthogonal transformation (2.2) is available [10].

The results on singular control are used to solve the singular LQ control problem of descriptor systems, under the constraints of impulse free, marginally stable and physically realizable closed loop system.

References

[1] D. J. Bender and A. J. Laub, "The linear quadratic optimal regulator for descriptor systems," *IEEE Trans. Autom. Control*, vol. AC-32, no. 8, pp. 672–688, Aug. 1987.

[2] D. J. Clements and K. Glover, "Spectral factorization via Hermitian pencils," *Linear Algebra Appl.*, vol. 122, no. 124, pp. 797–846, 1989.

[3] D. J. Clements, B. D. O. Anderson, and P. J. Moylan, "Matrix inequality solution to linear-quadratic singular control problem," *IEEE Trans. Autom. Control*, vol. AC-22, no. 1, pp. 55–57, Feb. 1977.

[4] B. R. Copeland and M. G. Safonov, "A zero compensation approach to singular $\mathcal{H}_2$ and $\mathcal{H}_\infty$ problems," *Int. J. Robust Nonlin. Control*, vol. 5, pp. 71–106, 1995.

[5] M. L. J. Hautus and L. M. Silverman, "System structure and singular control," *Linear Algebra Appl.*, vol. 50, pp. 369–402, Apr. 1983.

[6] V. L. Mehrmann, *The Autonomous Linear Quadratic Control Problem: Theory and Numerical Solution*. New York: Springer Verlag, 1991, vol. 163.

[7] J. B. Moore, "The singular solutions to a singular quadratic minimization problem," *Int. J. Control*, vol. 20, no. 3, pp. 383–393, 1974.

[8] P. J. Moylan and J. B. Moore, "Generalizations of singular optimal control," *Automatica*, vol. 7, pp. 591–598, 1971.

[9] J. Stefanovski, "LQ control of descriptor systems: A spectral factorisation approach," *Int. J. Control*, vol. 83, no. 3, pp. 585–600, 2010.

[10] The Polynomial Toolbox for MATLAB, Manual and Commands [Online]. Available: www.polyx.com

[11] J. C. Willems, A. Kitapci, and L. M. Silverman, "Singular optimal control: A geometric approach," *SIAM J. Control Optim.*, vol. 24, no. 2, pp. 323–337, Mar. 1986.

Compensating the Distributed Effect of Diffusion and Counter-Convection in Multi-Input and Multi-Output LTI Systems

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***Abstract*—Compensation of infinite-dimensional input or sensor dynamics in SISO, LTI systems is achieved using the backstepping method. For MIMO, LTI systems with distributed input or sensor dynamics, governed by diffusion with counter-convection, we develop a methodology for constructing control laws and observers that compensate the infinite-dimensional actuator or sensor dynamics. The explicit construction of the compensators are based on novel transformations, which can be considered of "backstepping-forwarding" type, of the finite-dimensional state of the plant and of the infinite-dimensional actuator or sensor states. Based on these transformations we construct explicit Lyapunov functionals which prove exponential stability of the closed-loop system, or convergence of the estimation error in the case of observer design. Finally, we illustrate the effectiveness of our controller with a numerical example.**

***Index Terms*—Multiple-input multiple-output (MIMO).**

I. Introduction

For LTI systems with input or sensor delays, predictor-based techniques have been very successful in control and observer design [1], [8], [9], [11], [12], [16]. Extensions of these techniques to nonlinear systems have been developed recently [4], [5], [7], while adaptive versions can be also found [2], [10].

The first efforts in designing predictor feedback controllers for realistic forms of infinite-dimensional actuator and sensor dynamics different than pure delays can be found in [7] and [14]. In [7] input and sensor dynamics governed by diffusion PDEs are compensated using the backstepping method for PDEs [6]. In [14] the results are extended to the case of input and sensor dynamics governed by diffusion with counter-convection. Finally, also in [7], the backstepping method is used for control and observer design in LTI systems with a string PDE in the actuation or sensing path. However, the applicability of the backstepping is limited to the class of systems where the ODE-PDE cascade is in the strict-feedback form.

In this work we consider the system

$$\dot{X}(t) = AX(t) + \int_0^D B(y)u(y,t)dy \tag{1}$$

$$u_t(x,t) = \epsilon u_{xx}(x,t) - bu_x(x,t) \tag{2}$$

$$u(D,t) = U(t) \tag{3}$$

where $X(t) \in \mathbb{R}^n$, $U(t) \in \mathbb{R}^m$, $D > 0$ and $x \in [0, D]$. The backstepping method is not applicable neither in the case of single-input ODE systems with distributed input nor in the case of multi-input ODE systems in which the diffusion and/or the counter-convection coefficients of the inputs's dynamics are different in each individual input channel. This is since the system that comprised of the finite-dimensional state $X(t)$ and the infinite-dimensional actuator states $u(x,t)$, $x \in [0, D]$, are not in the strict-feedback form. The controller design for these cases

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is recognized as one of the "key limitations" of the backstepping design [7], Chapter 1.10. In this technical note, novel forwarding-backstepping transformations of the finite-dimensional state of the plant and of the infinite-dimensional actuator states are introduced to transform the system to an exponentially stable system. These transformations completely decouple the ODE-PDE interconnection. By constructing a Lyapunov functional we prove exponential stability of the transformed system. The invertibility of our transformations guarantees the exponential stability of the system in the original variables.

In this technical note we consider the following special cases of (1)–(3).

1) Distributed Diffusion With Counter-Convection: In this case $\epsilon = 1$, $b > 0$ and there is an additional boundary condition in (1)–(3), $u_x(0,t) = 0$. Note here that a mixed-type condition, i.e., $u_x(0,t) = qu(0,t)$, can be treated analogously. Moreover, we consider a Neumann or a mixed-type condition to be the most challenging one, since in this case the actuator dynamics can be unstable. This is not the case with the actuator dynamics given by (2)–(3) and with a Dirichlet boundary condition, i.e., $u(0,t) = 0$. The term $-bu_x(x,t)$ has the effect of "counter-convection," namely an effect which opposes the propagation of the control signal $U(t)$ from $x = D$ to $x = 0$. We rely on the presence of diffusion in (2) to achieve stabilization of the (u, X) system in the presence of counter-convection. A practical example of this behavior can be a model of a plasma that flows towards the actuator and the actuator sends current upstream (see [13] and the references therein). This type of input dynamics are compensated in [14] for the special case of single-input systems, with $B(y) = B\delta(y)$, where $\delta(y)$ is the Dirac function. In this work we construct explicit feedback laws for multi-input systems with distributed inputs that enter the system through channels with different diffusion and counter-convection coefficients (Section II). We also develop a dual of our actuator dynamics compensator and design an infinite-dimensional observer which compensates the dynamics of the sensor (Section III). Finally, a numerical example of a second order single-input system, with distributed input dynamics governed by diffusion with counter-convection, demonstrates the effectiveness of the proposed controller (Section IV).

2) Distributed Diffusion: In this case $\epsilon = 1$, $b = 0$. An example of this behavior can be a linearized model for fluid-structure interaction where the ODE (that can be used to approximate an infinite-dimensional system) component has a distributed coupling with the heat component (see [15] and the references therein). This type of input dynamics are compensated in [7] for the special case of single-input systems, with $B(y) = B\delta(y)$ and $\delta(y)$ being the Dirac function. As in the previous case, compensators for both actuator and sensor dynamics (as special cases of the designs in Sections II and III, respectively) governed by diffusion are constructed (Remarks 2.1 and 3.1).

II. Controller Design for Distributed Diffusion With Counter-Convection

We consider the system

$$\dot{X}(t) = AX(t) + \sum_{i=1}^{2}\int_0^{D_i} B_i(y)u_i(y,t)dy \tag{4}$$
$$\partial_t u_1(x,t) = \partial_{xx}u_1(x,t) - b_1\partial_x u_1(x,t) \tag{5}$$
$$\partial_x u_1(0,t) = 0 \tag{6}$$
$$u_1(D_1,t) = U_1(t) \tag{7}$$
$$\partial_t u_2(z,t) = \partial_{zz}u_2(z,t) - b_2\partial_z u_2(z,t) \tag{8}$$
$$\partial_z u_2(0,t) = 0 \tag{9}$$
$$u_2(D_2,t) = U_2(t) \tag{10}$$

where $X(t) \in \mathbb{R}^n$, $U_1(t), U_2(t) \in \mathbb{R}$, $x \in [0, D_1]$, $z \in [0, D_2]$ and $b_1, b_2 > 0$. For notational simplicity we consider a two-input case. The same analysis can be carried out for an arbitrary number of inputs. We next state the main result of this section.

Theorem 1: Consider the closed-loop system consisting of the plant (4)–(10) and the control law

$$\begin{aligned} U(t) &= \begin{bmatrix} U_1(t) \\ U_2(t) \end{bmatrix} \\ &= \begin{bmatrix} K_1Z(t) - \frac{b_1}{2}\int_0^{D_1} e^{b_1(D_1-y)}u_1(y,t)dy \\ K_2Z(t) - \frac{b_2}{2}\int_0^{D_2} e^{b_2(D_2-y)}u_2(y,t)dy \end{bmatrix} \end{aligned} \tag{11}$$

$$Z(t) = X(t) + \sum_{i=1}^{2}\int_0^{D_i} g_i(y)u_i(y,t)dy \tag{12}$$

$$\begin{aligned} g_i(x) = [I \quad 0]&\left(e^{A_ix}\begin{bmatrix} I \\ -b_iI \end{bmatrix} - \int_0^x e^{A_i(x-y)}\right.\\ &\left.\times \begin{bmatrix} 0 \\ I \end{bmatrix}\frac{b_i}{2}e^{b_i(D_i-y)}dyR_i\right)F_i - [I \quad 0]\\ &\times\left(\int_0^x e^{A_i(x-y)}\begin{bmatrix} 0 \\ I \end{bmatrix}B_i(y)dy + \int_0^x e^{A_i(x-y)}\right.\\ &\left.\times\begin{bmatrix} 0 \\ I \end{bmatrix}\frac{b_i}{2}e^{b_i(D_i-y)}dyG_i\right) \end{aligned} \tag{13}$$

$$A_i = \begin{bmatrix} 0 & I \\ A & -b_iI \end{bmatrix} \tag{14}$$

$$M_i = I + \int_0^{D_i}[0 \quad I]\,e^{A_i(D_i-y)}\begin{bmatrix} 0 \\ I \end{bmatrix}\frac{b_i}{2}e^{b_i(D_i-y)}dy \tag{15}$$

$$R_i = M_i^{-1}[0 \quad I]\,e^{A_iD_i}\begin{bmatrix} I \\ -b_iI \end{bmatrix} \tag{16}$$

$$G_i = -M_i^{-1}\int_0^{D_i}[0 \quad I]\,e^{A_i(D_i-y)}\begin{bmatrix} 0 \\ I \end{bmatrix}B_i(y)dy \tag{17}$$

$$\begin{aligned} E_i = [I \quad 0]\,e^{A_iD_i}&\begin{bmatrix} I \\ -b_iI \end{bmatrix} - \int_0^{D_i}[I \quad 0]\\ &\times e^{A_i(D_i-y)}\begin{bmatrix} 0 \\ I \end{bmatrix}\frac{b_i}{2}e^{b_i(D_i-y)}dyR_i \end{aligned} \tag{18}$$

$$\begin{aligned} F_i = E_i^{-1}&\left(\int_0^{D_i}[I \quad 0]\,e^{A_i(D_i-y)}\begin{bmatrix} 0 \\ I \end{bmatrix}B_i(y)dy\right.\\ &+\int_0^{D_i}[I \quad 0]\,e^{A_i(D_i-y)}\begin{bmatrix} 0 \\ I \end{bmatrix}\frac{b_i}{2}e^{b_i(D_i-y)}\\ &\left.\times dyG_i\right), \quad i = 1,2 \end{aligned} \tag{19}$$

and the block matrices I and 0 are of dimension $n \times n$. Let the pair $(A, [\,g_1'(D_1) \quad g_2'(D_2)\,])$, be completely controllable and let the matrices M_i, R_i, $i = 1, 2$ be invertible. Moreover, choose K_1 and K_2 such that the matrix $A_{cl} = A - g_1'(D_1)K_1 - g_2'(D_2)K_2$ is Hurwitz and such that none of its eigenvalues are located at $-(b_i^2/4) - (2\kappa+1)^2\pi^2/4D^2, i = 1, 2$. Then for any initial conditions $u_i(\cdot, 0) \in L^2(0, D_i)$, $i = 1, 2$ the closed-loop system has a unique solution $(X(t), u_1(\cdot,t), u_2(\cdot,t),) \in C([0,\infty], \mathbb{R}^n \times L^2(0,D_1) \times L^2(0,D_2))$ which is exponentially stable in the sense that there exist positive constants η and ν such that

$$\Omega(t) \leq \eta\Omega(0)e^{-\nu t} \tag{20}$$

$$\begin{aligned} \Omega(t) = |X(t)|^2 &+ \int_0^{D_1} u_1(x,t)^2dx \\ &+ \int_0^{D_2} u_2(z,t)^2dz. \end{aligned} \tag{21}$$

Proof: Consider the transformation of the finite-dimensional state $X(t), Z(t)$ given in (12) and the transformations of the infinite-dimensional actuator states $u_1(x,t)$ and $u_2(z,t)$ given by

$$\begin{aligned} w_1(x,t) =& u_1(x,t) - \gamma_1(x) \\ &\times \left(X(t) + \sum_{i=1}^{2} \int_0^{D_i} g_i(y) \times u_i(y,t)dy \right) \\ &+ \frac{b_1}{2} \int_0^x e^{b_1(x-y)} u_1(y,t)dy \end{aligned} \tag{22}$$

$$\begin{aligned} w_2(z,t) =& u_2(z,t) - \gamma_2(z) \\ &\times \left(X(t) + \sum_{i=1}^{2} \int_0^{D_i} g_i(y) \times u_i(y,t)dy \right) \\ &+ \frac{b_2}{2} \int_0^z e^{b_2(z-y)} u_2(y,t)dy \end{aligned} \tag{23}$$

where the kernels $\gamma_i(\cdot)$, $i = 1, 2$ are to be specified. Using (12) and (22)–(23) we map the plant (4)–(10) to the "target system"

$$\dot{Z}(t) = A_{cl} Z(t) \tag{24}$$
$$\partial_t w_1(x,t) = \partial_{xx} w_1(x,t) - b_1 \partial_x w_1(x,t) \tag{25}$$
$$\partial_x w_1(0,t) = \frac{b_1}{2} w_1(0,t) \tag{26}$$
$$w_1(D_1,t) = 0 \tag{27}$$
$$\partial_t w_2(z,t) = \partial_{zz} w_2(z,t) - b_2 \partial_z w_2(z,t) \tag{28}$$
$$\partial_z w_2(0,t) = \frac{b_2}{2} w_2(0,t) \tag{29}$$
$$w_2(D_2,t) = 0. \tag{30}$$

To see this we first differentiate $Z(t)$ in (12). Using relations (5) and (8), integration by parts and relations (6)–(7), (9)–(10) and (12) we get

$$\begin{aligned} \dot{Z}(t) =& AZ(t) + \sum_{i=1}^{2} \Bigg(\int_0^{D_i} \Bigg(-Ag_i(y) + B_i(y) + b_i g_i'(y) \\ &+ g_i''(y) + \left(g_i'(D_i) + b_i g_i(D_i) \right) \frac{b_i}{2} e^{b_i(D_i-y)} \Bigg) \\ &\times u_i(y,t)dy + g_i(D_i)\partial_y u_i(D_i,t) + \left(g_i'(0) + b_i g_i(0) \right) \\ &\times u_i(0,t) - \left(g_i'(D_i) + b_i g_i(D_i) \right) \Bigg(U_i(t) \\ &+ \int_0^{D_i} \frac{b_i}{2} e^{b_i(D_i-y)} u_i(y,t)dy \Bigg) \Bigg). \end{aligned} \tag{31}$$

Since the $g_i(\cdot)$, $i = 1, 2$ in (13) are the solutions of the following boundary value problems

$$\begin{aligned} g_i''(r) =& -b_i g_i'(r) + Ag_i(r) - B_i(r) \\ &- \left(g_i'(D_i) + b_i g_i(D_i) \right) \frac{b_i}{2} e^{b_i(D_i-r)} \end{aligned} \tag{32}$$
$$g_i(D_i) = 0 \tag{33}$$
$$g_i'(0) = -b_i g_i(0), \quad i = 1, 2 \tag{34}$$

relation (31) becomes $\dot{Z}(t) = AZ(t) - \sum_{i=1}^{2} g_i'(D_i) \left(U_i(t) + (b_i/2) \int_0^{D_i} e^{b_i(D_i-y)} u_i(y,t)dy \right)$. Therefore, with the controller (11) we take (24). We now prove (25)–(27) (the proof of (28)–(30) follows exactly the same pattern). Differentiating (22) with respect to time, using integration by parts together with relations (4)–(10) and using the fact that $\partial_t \left(X(t) + \int_0^{D_1} g_1(y)u_1(y,t)dy + \int_0^{D_2} g_2(y)u_2(y,t)dy \right) = \dot{Z}(t) = A_{cl}Z(t)$, we get

$$\begin{aligned} \partial_t w_1(x,t) =& \partial_{xx} u_1(x,t) \\ &- b_1 \partial_x u_1(x,t) - \gamma_1(x) A_{cl} \\ &\times \left(X(t) + \sum_{i=1}^{2} \int_0^{D_i} g_i(y) u_i(y,t)dy \right) \\ &+ \frac{b_1}{2} \partial_x u_1(x,t). \end{aligned} \tag{35}$$

By taking the spatial derivatives of (22) and if the kernels $\gamma_i(\cdot)$ satisfy

$$\gamma_i''(r) = \gamma_i(r) A_{cl} + b_i \gamma_i'(r) \tag{36}$$
$$\gamma_i(D_i) = K_i \tag{37}$$
$$\gamma_i'(0) = \frac{b_i}{2} \gamma_i(0), \quad i = 1, 2 \tag{38}$$

i.e.

$$\gamma_1(x) = K_1 \Gamma_1 \left[I \quad \tfrac{b_1}{2} I \right] e^{\begin{bmatrix} 0 & A_{cl} \\ I & b_1 I \end{bmatrix} x} \begin{bmatrix} I \\ 0 \end{bmatrix} \tag{39}$$
$$\gamma_2(z) = K_2 \Gamma_2 \left[I \quad \tfrac{b_2}{2} I \right] e^{\begin{bmatrix} 0 & A_{cl} \\ I & b_2 I \end{bmatrix} z} \begin{bmatrix} I \\ 0 \end{bmatrix} \tag{40}$$
$$\Gamma_i = \Lambda_i^{-1}, \quad i = 1, 2 \tag{41}$$
$$\begin{aligned} \Lambda_i =& \left[I \quad \tfrac{b_i}{2} I \right] e^{\begin{bmatrix} 0 & A_{cl} \\ I & b_i I \end{bmatrix} D_i} \begin{bmatrix} I \\ 0 \end{bmatrix}, \\ & i = 1, 2 \end{aligned} \tag{42}$$

then, combining (35) with the spatial derivatives of (22) and (36), we get (25). Moreover, by setting $x = 0$ in (22) and in the first spatial derivative of (22) and by taking into account (6), (38) we arrive at (26). Finally, by setting $x = D$ in (22) and using relations (11) and (37) we get (27).

The matrices Λ_i, $i = 1, 2$ can be shown to be invertible as follows: Using the Maclaurin series expansion of the exponential matrix in the expressions for Λ_i, $i = 1, 2$ and matching the terms for the powers of D_i, $i = 1, 2$ one can conclude that the following holds $\Lambda_i = \sum_{m=0}^{\infty} \left((b_i^2/4)I + A_{cl} \right)^m D_i^{2m}/2m! \sum_{k=0}^{\infty} (b_i/2)^k D_i^k/k! = e^{(b_i/2)D_i} \sum_{m=0}^{\infty} \left((b_i^2/4)I + A_{cl} \right)^m (D_i^{2m}/2m!)$. Assume now a Jordan form for the matrix $(b_i^2/4)I + A_{cl}$ as $b_i^2/4I + A_{cl} = M \mathrm{diag}\,(J_1, \ldots, J_r)\, M^{-1}$, where each Jordan block J_k for $k = 1, \ldots, r$ corresponds to an eigenvalue λ_k of the matrix $(b_i^2/4)I + A_{cl}$. Then the Λ_i matrices are invertible if and only if the matrix $M\Lambda_i M^{-1}$ is invertible. Hence, we have $M\Lambda_i M^{-1} = e^{b_i/2D_i} M \sum_{m=0}^{\infty} \left((b_i^2/4)I + A_{cl} \right)^m D_i^{2m}/2m! M^{-1} = e^{b_i/2D_i} \sum_{m=0}^{\infty} \left[M \left(b_i^2/4I + A_{cl} \right) M^{-1} \right]^m D_i^{2m}/2m!$ and thus $M\Lambda_i M^{-1} = e^{b_i/2D_i} \mathrm{diag}\left(\sum_{m=0}^{\infty} J_1^m D_i^{2m}/2m!, \ldots, \sum_{m=0}^{\infty} J_r^m D_i^{2m}/2m! \right)$. Since the blocks J_k are upper diagonal and since $e^{(b_i/2)D_i}$ is different than zero, we can conclude that the Λ_i are invertible if it holds that $\cosh\left(\sqrt{\lambda_k} D \right) = \sum_{m=0}^{\infty} \left(\lambda_k D_i^2 \right)^m / 2m! \neq 0$, for all $k = 1, \ldots, r$, which is true if $\lambda_k \neq -(2\kappa + 1)^2 \pi^2 / 4D^2$ for all $k = 1, \ldots, r$ and κ is a natural number. To sum-up, both the conditions of stability of A_{cl} and the invertibility of the matrices Λ_i are satisfied if we choose K_1 and K_2 such that A_{cl} is Hurwitz and none of the eigenvalues of the matrix $(b_i^2/4)I + A_{cl}$ are located at the positions $-(2\kappa + 1)^2 \pi^2 / 4D^2$. This is always possible provided that the controllability conditions are satisfied. Similarly one obtains (28)–(30).

In a similar manner the inverse transformations of (22)–(23) are given by

$$\begin{aligned} u_1(x,t) &= w_1(x,t) + \delta_1(x)Z(t) \\ &\quad - \frac{b_1}{2}\int_0^x e^{(b_1/2)(x-y)} w_1(y,t)dy \qquad (43) \\ u_2(z,t) &= w_2(z,t) + \delta_2(z)Z(t) \\ &\quad - \frac{b_2}{2}\int_0^z e^{(b_2/2)(z-y)} w_2(y,t)dy \qquad (44) \end{aligned}$$

where the $\delta_i(\cdot)$ satisfy $\delta_i''(r) = \delta_i(r)A_{cl} + b_i\delta_i'(r)$ with boundary conditions as $\delta_i(0) = K_i\Gamma_i$ and $\delta_i'(0) = 0$, $i = 1, 2$, i.e.

$$\delta_i(w) = K_i\Gamma_i [I \quad 0] e^{\begin{bmatrix} 0 & A_{cl} \\ I & b_i I \end{bmatrix} w} \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad i = 1, 2. \qquad (45)$$

Finally, the inverse transformation of (12) is given by

$$X(t) = \left(I - \sum_{i=1}^{2}\int_0^{D_i} g_i(y)\delta_i(y)\right) Z(t) - \sum_{i=1}^{2}\int_0^{D_i} g_i(y) \times \left(w_i(y,t) - \frac{b_i}{2}\int_0^y e^{(b_i/2)(y-r)} w_i(r,t)dr\right)dy. \qquad (46)$$

Consider now the Lyapunov function $V(t) = Z(t)^T PZ(t) + (1/2)\int_0^{D_1} w_1(x,t)^2dx + (1/2)\int_0^{D_2} w_2(z,t)^2dz$, where $P = P^T > 0$ and $Q = Q^T > 0$ satisfy

$$\left(A - g_1'(D_1)K_1 - g_2'(D_2)K_2\right)^T P + P\left(A - g_1'(D_1)K_1 - g_2'(D_2)K_2\right) = -Q. \qquad (47)$$

With the help of (25), (28), (47) and using integration by parts we obtain

$$\begin{aligned} \dot{V}(t) \le &- \lambda_{\min}(Q)\left|Z(t)\right|^2 + w_1(x,t)\partial_x w_1(x,t)\Big|_{x=0}^{x=D_1} \\ &- \int_0^{D_1} \partial_x w_1(x,t)^2dx - \frac{b_1}{2}\, w_1(x,t)^2\Big|_{x=0}^{x=D_1} \\ &+ w_2(z,t)\partial_z w_2(z,t)\Big|_{z=0}^{z=D_2} - \int_0^{D_2} \partial_z w_2(z,t)^2dz \\ &- \frac{b_2}{2}\, w_2(z,t)^2\Big|_{z=0}^{z=D_2} \end{aligned} \qquad (48)$$

where $\lambda_{\min}(Q)$ is the smallest eigenvalue of Q in (47). Using the boundary conditions (26)–(27), (29)–(30) and the Poincare inequality we obtain $\dot{V}(t) \le -\lambda_{\min}(Q)\left|Z(t)\right|^2 - (1/4D_1^2)\int_0^{D_1} w_1(x,t)^2dx - (1/4D_2^2)\int_0^{D_2} w_2(z,t)^2dz$. Thus, if we set $\rho = \min\left\{\lambda_{\min}(Q)/2\lambda_{\max}(P), 1/2D_1^2, 1/2D_2^2\right\}$ and $\lambda_{\max}(P)$ is the largest eigenvalue of P in (47) we obtain $\dot{V}(t) \le -\rho V(t)$. Using the comparison principle we get $V(t) \le V(0)e^{-\rho t}$. To prove stability in the original variables $X(t)$ and $u_1(x,t)$, $u_2(z,t)$ it is sufficient to show that

$$\underline{M}\Omega(t) \le V(t) \le \overline{M}\Omega(t) \qquad (49)$$

for some positive $\overline{M}$ and $\underline{M}$ and Theorem 1 is proved with $\eta = \overline{M}/\underline{M}$. Is straightforward to establish bound (49) using expressions (12), (22)–(23) and (43)–(44), (46) together with Young and Cauchy-Schwartz's inequalities. Bound (49) holds with $\overline{M} = \left(3\omega + \sum_{i=1}^2 3\left(1 + D_i^2(b_i^2/4)e^{2b_iD_i}\right) + 9D_i\sup_{y\in[0,D_i]}\gamma_i(y)^2\omega\right)r$ where $r = \max\{\lambda_{\max}(P), 1/2\}$, $\underline{M} = \min\{\lambda_{\min}(P), 1/2\}/M$, $\omega = 1 + \sum_{i=1}^2 D_i\sup_{y\in[0,D_i]} g_i(y)^2$, $M = 9\left(1 + \sum_{i=1}^2 D_i\sup_{y\in[0,D_i]} g_i(y)^2\delta_i(y)^2\right) + 6\sum_{i=1}^2 D_i\sup_{y\in[0,D_i]} g_i(y)^2\left(1 + D_i(b_i^2/4)e^{bD_i}\right) + \sum_{i=1}^2\left(3\left(1 + D_i^2(b_i^2/4)e^{b_iD_i}\right) + 9D_i\sup_{y\in[0,D_i]}\delta_i(y)^2\right)$. Moreover, from (24) we conclude that $Z(t)$ is bounded and converges exponentially to zero. From (22)–(23) one can conclude that $w_i(\cdot,0) \in L^2(0,D_i)$, $i = 1, 2$ and thus it follows from (25)–(30) that $w_i(\cdot,t) \in C(L^2(0,D_i))$, $i = 1, 2$. Using the inverse transformations (43)–(44) we can conclude that $u_i(\cdot,t) \in C(L^2(0,D_i))$, $i = 1, 2$. The uniqueness of weak solution is proved using the uniqueness of weak solution to the boundary problems (25)–(30) (see, e.g., [3]). Hence, the theorem is proved. ■

Remark 2.1: An interesting special case of the system (4)–(10) is the case where $b_i = 0$, $i = 1, 2$. That is, the inputs to the plant satisfy diffusion equations. In this case Theorem 1 applies by setting $b_i = 0$, $i = 1, 2$ in (11)–(19). It is important here to observe that the direct transformations (22)–(23) are significantly simplified when we set $b_i = 0$, $i = 1, 2$. Moreover, the inverse transformations (43)–(44) are trivially satisfied with $\delta_i(\cdot) = \gamma_i(\cdot)$, $i = 1, 2$. One can see this by looking at the expressions (39)–(40) and (45) when $b_i = 0$, $i = 1, 2$, or by observing that (22)–(23) when $b_i = 0$, $i = 1, 2$, can be written as $w_1(x,t) = u_1(x,t) - \gamma_1(x)Z(t)$ and $w_2(z,t) = u_2(z,t) - \gamma_2(z)Z(t)$.

Remark 2.2: We consider the case of a Neumann or a mixed-type condition to be the most challenging one, since in this case the actuator dynamics can be unstable. In the case of a Dirichlet boundary condition, i.e., $u_i(0,t) = 0$, $i = 1, 2$ then the boundary condition of the target system can be chosen as $w_i(0,t) = 0$, $i = 1, 2$ since the actuator dynamics are stable for $U_i(t) = 0$, $i = 1, 2$. The case of a mixed type condition can be treated very similarly with the case of a Neumann boundary condition. Consider for example a boundary condition as $\partial_x u_i(0,t) = q_iu_i(0,t)$, $i = 1, 2$. Then it can be shown that the boundary value problem (36)–(38) will remain the same whereas the problem (32)–(34) will be as $g_i''(r) = -b_ig_i'(r) + Ag_i(r) - B_i(r) - g_i'(D_i)\left(b_i/2 - q_i\right)e^{(b_i+q_i)(D_i-r)}$ with boundary conditions as $g_i(D_i) = 0$ and $g_i'(0) = -(b_i + q_i)g_i(0)$, $i = 1, 2$. In the case of a Dirichlet boundary condition the control law and consequently the transformations $w_i(x,t)$, $i = 1, 2$ are significantly simplified. Specifically, the control laws simply become $U_i(t) = K_iZ(t)$, $i = 1, 2$ and the $w_i(x,t)$, $i = 1, 2$ transformations become $w_i(x,t) = u_i(x,t) - \gamma_i(x)\left(X(t) + \int_0^{D_1} g_1(y)u_1(y,t)dy + \int_0^{D_2} g_2(y)u_2(y,t)dy\right)$, where $\gamma_i''(r) = \gamma_i(r)A_{cl} + b_i\gamma_i'(r)$, $\gamma_i(D_i) = K_i$ and $\gamma_i(0) = 0$, $i = 1, 2$.

III. Observer Design With Distributed Diffusion With Counter-Convection

In this section, we consider the system

$$\begin{aligned} \dot{X}(t) &= AX(t) + BU(t) && (50) \\ \partial_t\xi_1(x,t) &= \partial_{xx}\xi_1(x,t) - b_1\partial_x\xi_1(x,t) + C_1(x)X(t) && (51) \\ \partial_x\xi_1(0,t) &= 0 && (52) \\ \xi_1(D_1,t) &= 0 && (53) \\ \partial_t\xi_2(z,t) &= \partial_{zz}\xi_2(z,t) - b_2\partial_z\xi_2(z,t) + C_2(z)X(t) && (54) \\ \partial_z\xi_2(0,t) &= 0 && (55) \\ \xi_2(D_2,t) &= 0 && (56) \\ Y_1(t) &= \xi_1(0,t) && (57) \\ Y_2(t) &= \xi_2(0,t). && (58) \end{aligned}$$

Next we state a new observer that compensates the sensor dynamics and prove exponential convergence of the resulting observer error system.

Theorem 2: Let the pair $\left(A, \begin{bmatrix} \gamma_1(0) \\ \gamma_2(0) \end{bmatrix}\right)$ be observable, where

$$\gamma_1(x) = \Gamma_1 \left[I \quad \tfrac{b_1}{2} I \right] e^{A_1 x} \begin{bmatrix} I \\ 0 \end{bmatrix} - \int_0^x \left[0 \quad C_1(y) \right] e^{A_1(x-y)} \begin{bmatrix} I \\ 0 \end{bmatrix} dy \tag{59}$$

$$\gamma_2(z) = \Gamma_2 \left[I \quad \tfrac{b_2}{2} I \right] e^{A_2 z} \begin{bmatrix} I \\ 0 \end{bmatrix} - \int_0^z \left[0 \quad C_2(y) \right] e^{A_2(z-y)} \begin{bmatrix} I \\ 0 \end{bmatrix} dy \tag{60}$$

$$A_i = \begin{bmatrix} 0 & A \\ I & b_i I \end{bmatrix} \tag{61}$$

$$N_i = \left[I \quad \tfrac{b_i}{2} I \right] e^{A_i D_i} \begin{bmatrix} I \\ 0 \end{bmatrix} \tag{62}$$

$$\Gamma_i = \int_0^{D_i} \left[0 \quad C_i(y) \right] e^{A_i(D_i-y)} \begin{bmatrix} I \\ 0 \end{bmatrix} dy N_i^{-1}, \quad i = 1, 2. \tag{63}$$

Let the matrices N_i, $i = 1, 2$ be invertible, and define the observer

$$\dot{\hat{X}}(t) = A\hat{X}(t) + BU(t) + \sum_{i=1}^{2} L_i \left(Y_i(t) - \hat{Y}_i(t) \right) \tag{64}$$

$$\partial_t \hat{\xi}_1(x,t) = \partial_{xx} \hat{\xi}_1(x,t) - b_1 \partial_x \hat{\xi}_1(x,t) + C_1(x)\hat{X}(t) + \gamma_1(x) \sum_{i=1}^{2} L_i \times \left(Y_i(t) - \hat{Y}_i(t) \right) \tag{65}$$

$$\partial_x \hat{\xi}_1(0,t) = -\frac{b_1}{2} \left(Y_1(t) - \hat{Y}_1(t) \right) \tag{66}$$

$$\hat{\xi}_1(D_1,t) = 0 \tag{67}$$

$$\partial_t \hat{\xi}_2(z,t) = \partial_{zz} \hat{\xi}_2(z,t) - b_2 \partial_z \hat{\xi}_2(z,t) + C_2(z)\hat{X}(t) + \gamma_2(z) \sum_{i=1}^{2} L_i \left(Y_i(t) - \hat{Y}_i(t) \right) \tag{68}$$

$$\partial_z \hat{\xi}_2(0,t) = -\frac{b_2}{2} \left(Y_2(t) - \hat{Y}_2(t) \right) \tag{69}$$

$$\hat{\xi}_2(D_2,t) = 0 \tag{70}$$

$$\hat{Y}_1(t) = \hat{\xi}_1(0,t) \tag{71}$$

$$\hat{Y}_2(t) = \hat{\xi}(0,t) \tag{72}$$

where L_1 and L_2 are chosen such that $A - L_1\gamma_1(0) - L_2\gamma_2(0)$ is Hurwitz. Then for any $(\xi_i(\cdot,0), \hat{\xi}_i(\cdot,0)) \in L^2(0,D_i)$, $i = 1$, 2 the observer error system has a unique solution $\left(X(t) - \hat{X}(t), \xi_1(\cdot,t) - \hat{\xi}_1(\cdot,t), \xi_2(\cdot,t) - \hat{\xi}_2(\cdot,t)\right) \in C([0,\infty], \mathbb{R}^n \times L^2(0,D_1) \times L^2(0,D_2))$ which is exponentially stable in the sense that there exist positive constants κ and λ such that

$$\Xi(t) \le \kappa \Xi(0) e^{-\lambda t} \tag{73}$$

$$\Xi(t) = \left| X(t) - \hat{X}(t) \right|^2 + \int_0^{D_1} \left(\xi_1(x,t) - \hat{\xi}_1(x,t) \right)^2 dx + \int_0^{D_2} \left(\xi_2(z,t) - \hat{\xi}_2(z,t) \right)^2 dz. \tag{74}$$

Proof: Introducing the error variables $\tilde{X}(t) = X(t) - \hat{X}(t)$, $\tilde{\xi}_1(x,t) = \xi_1(x,t) - \hat{\xi}_1(x,t)$ and $\tilde{\xi}_2(z,t) = \xi_2(z,t) - \hat{\xi}_2(z,t)$, we obtain $\dot{\tilde{X}}(t) = A\tilde{X}(t) - L_1\tilde{\xi}_1(0,t) - L_2\tilde{\xi}_2(0,t)$, $\partial_t \tilde{\xi}_1(x,t) = \partial_{xx}\tilde{\xi}_1(x,t) - b_1\partial_x\tilde{\xi}_1(x,t) + C_1(x)\tilde{X}(t) - \gamma_1(x)L_1\tilde{\xi}_1(0,t) - \gamma_1(x)L_2\tilde{\xi}_2(0,t)$, $\partial_x\tilde{\xi}_1(0,t) = b_1/2\tilde{\xi}_1(0,t)$, $\tilde{\xi}_1(D_1,t) = 0$ and $\partial_t\tilde{\xi}_2(z,t) = \partial_{zz}\tilde{\xi}_2(z,t) - b_2\partial_z\tilde{\xi}_2(z,t) + C_2(z)\tilde{X}(t) - \gamma_2(z)L_1\tilde{\xi}_1(0,t) - \gamma_2(z)L_2\tilde{\xi}_2(0,t)$, $\partial_z\tilde{\xi}_2(0,t) = (b_2/2)\tilde{\xi}_2(0,t)$, $\tilde{\xi}_2(D_2,t) = 0$. With the transformations

$$\tilde{\zeta}_1(x,t) = \tilde{\xi}_1(x,t) - \gamma_1(x)\tilde{X}(t) \tag{75}$$

$$\tilde{\zeta}_2(z,t) = \tilde{\xi}_2(z,t) - \gamma_2(z)\tilde{X}(t), \tag{76}$$

and by noting that $\gamma_1(x)$ and $\gamma_2(z)$ in (59)–(60) are the solutions of the following boundary value problems $\gamma_i''(r) = \gamma_i(r)A + b_i\gamma_i'(r) - C_i(r)$, $\gamma_i(D_i) = 0$ and $\gamma_i'(0) = (b_i/2)\gamma_i(0)$, $i = 1, 2$, we get

$$\dot{\tilde{X}}(t) = \left(A - L_1\gamma_1(0) - L_2\gamma_2(0)\right)\tilde{X}(t) - L_1\tilde{\zeta}_1(0,t) - L_2\tilde{\zeta}_2(0,t) \tag{77}$$

$$\partial_t\tilde{\zeta}_1(x,t) = \partial_{xx}\tilde{\zeta}_1(x,t) - b_1\partial_x\tilde{\zeta}_1(x,t) \tag{78}$$

$$\partial_x\tilde{\zeta}_1(0,t) = \frac{b_1}{2}\tilde{\zeta}_1(0,t) \tag{79}$$

$$\tilde{\zeta}_1(D_1,t) = 0 \tag{80}$$

$$\partial_t\tilde{\zeta}_2(z,t) = \partial_{zz}\tilde{\zeta}_2(z,t) - b_2\partial_z\tilde{\zeta}_2(z,t) \tag{81}$$

$$\partial_z\tilde{\zeta}_2(0,t) = \frac{b_2}{2}\tilde{\zeta}_2(0,t) \tag{82}$$

$$\tilde{\zeta}_2(D_2,t) = 0. \tag{83}$$

Since the pair $\left(A, \begin{bmatrix} \gamma_1(0) \\ \gamma_2(0) \end{bmatrix}\right)$ is observable, we can choose L_1 and L_2 such that $\left(A - L_1\gamma_1(0) - L_2\gamma_2(0)\right)^T P + P\left(A - L_1\gamma_1(0) - L_2\gamma_2(0)\right) = -Q$. To establish exponential stability of the error system we use the following Lyapunov functional $V(t) = \tilde{X}^T(t)P\tilde{X}(t) + (\eta/2)\int_0^{D_1}\tilde{\zeta}_1^2(x,t)dx + (\nu/2)\int_0^{D_2}\tilde{\zeta}_2^2(z,t)dz$. Taking its time derivative of $V(t)$, using integration by parts, relations (77)–(83) and Young's inequality, similarly to the calculations in Section II, we get $\dot{V}(t) \le -\lambda_{\min}(Q)/2\left|\tilde{X}(t)\right|^2 - \eta\int_0^{D_1}\partial_x\tilde{\zeta}_1(x,t)^2dx - \nu\int_0^{D_2}\partial_z\tilde{\zeta}_2(z,t)^2dz + 4|P|^2|L_1|^2/\lambda_{\min}(Q)\zeta_1(0,t)^2 + 4|P|^2|L_2|^2/\lambda_{\min}(Q)\zeta_2(0,t)^2$. Using Poincare and Agmon's inequalities we get $\dot{V}(t) \le -\lambda_{\min}(Q)/2\left|\tilde{X}(t)\right|^2 - (\eta - m_1)\int_0^{D_1}\partial_x\tilde{\zeta}_1(x,t)^2dx - (\nu - m_2)\int_0^{D_2}\partial_z\tilde{\zeta}_2(z,t)^2dz$, with $m_1 = 16D_1|P|^2|L_1|^2/\lambda_{\min}(Q)$ and $m_2 = 16D_2|P|^2|L_2|^2/\lambda_{\min}(Q)$. Choosing $\eta = 2m_1$ and $\nu = 2m_2$ and using one more time Poincare's inequality we have $\dot{V}(t) \le -\sigma V(t)$, where $\sigma = \min\{\lambda_{\min}(Q)/2\lambda_{\max}(P), 1\}$. Now, as in the proof of Theorem 1, using (75)–(76) one can show that $\underline{M}\Xi(t) \le V(t) \le \overline{M}\Xi(t)$ with $\underline{M} = \min\{\lambda_{\min}(P), \eta/2, \nu/2\}/\ 3 + 2\sum_{i=1}^{2} D_i \sup_{y\in[0,D_i]}\gamma_i(y)^2$ and $\overline{M} = \left(3 + 2\sum_{i=1}^{2} D_i \sup_{y\in[0,D_i]}\gamma_i(y)^2\right)\max\{\lambda_{\max}(P), \eta/2, \nu/2\}$. From (75)–(76) one can conclude that $\tilde{\zeta}_i(\cdot,0) \in L^2(0,D_i)$, $i = 1, 2$ and thus it follows from (78)–(83) that $\tilde{\zeta}_i(\cdot,t) \in C(L^2(0,D_i))$, $i = 1$, 2. Thus from (77) it follows that $\tilde{X}(t)$ is bounded. Using the inverse transformations of (78)–(83) we can conclude that $\tilde{\xi}_i(\cdot,t) \in C(L^2(0,D_i))$, $i = 1, 2$. The uniqueness of weak solution is proved using the uniqueness of weak solution to the boundary problems (78)–(83) (see, e.g., [3]). This completes the proof. ■

Remark 3.1: As in Section II, setting $b_i = 0$, $i = 1, 2$ in relations (64)–(72) we recover the observer for a system with distributed sensor dynamics governed by diffusion, i.e., for the special case of system (50)–(58), with $b_i = 0$, $i = 1, 2$.

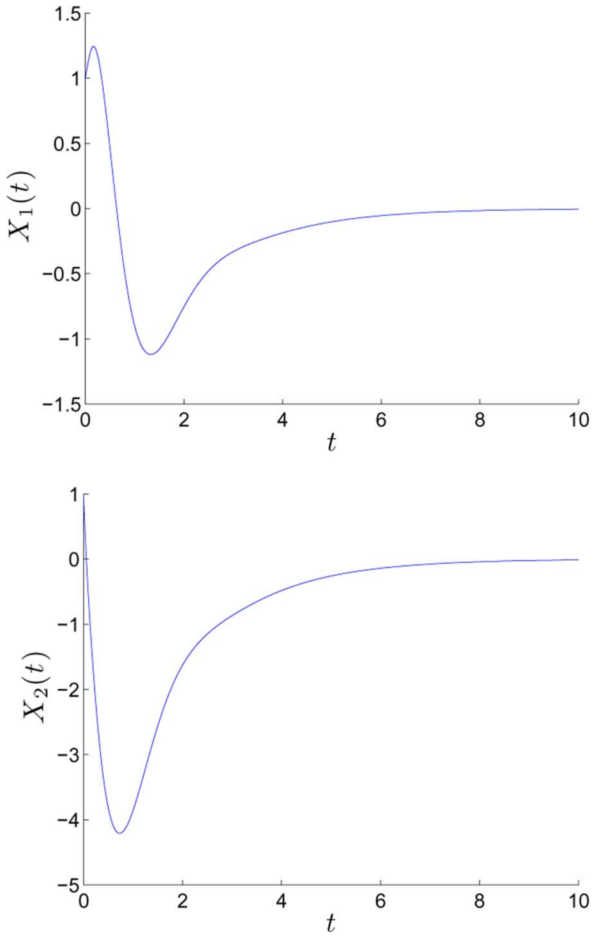

Fig. 1. Response of the system in the example with initial conditions $X_1(0) = X_2(0) = 1$.

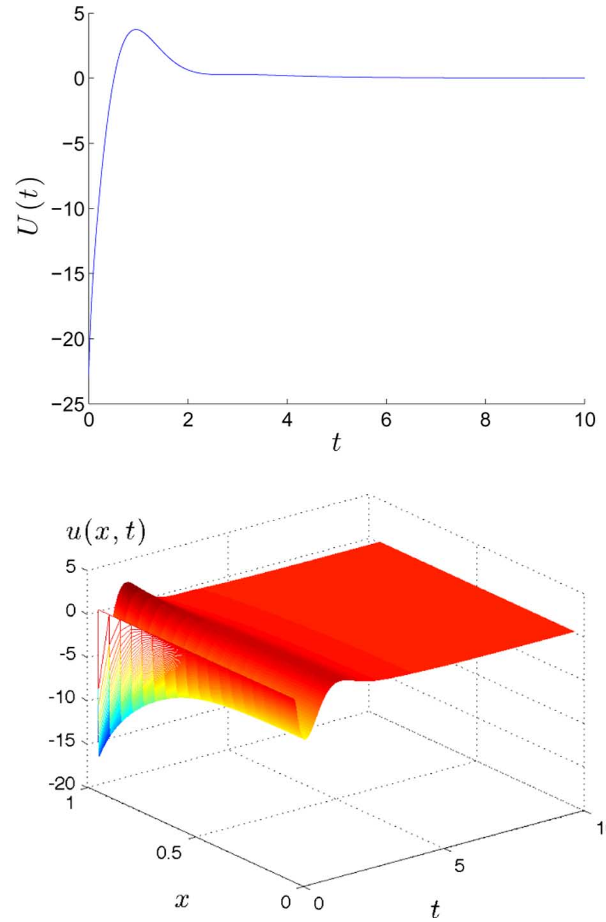

Fig. 2. Control effort (top figure) and the function $u(x,t)$ (bottom figure) of the system in the example with initial condition $u(x,0) = 1, \forall\, x \in [0,1)$.

IV. Example

We consider the system $\dot{X}(t) = AX(t) + B_0 u(0,t) + B_1 u(D,t)$, $\partial_t u(x,t) = \partial_{xx} u(x,t) - b\partial_x u(x,t)$, $\partial_x u(0,t) = 0$, $\partial_x u(0,t) = 0$ and $u(D,t) = U(t)$ where $x \in [0,D]$. This example is a special case of system (4) with one input and, say, $B(y) = B_0\delta(y) + B_1\delta(D-y)$, with $\delta(y)$ being the Dirac function. We choose the parameters of the system as $D = 1, b = 1, A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, B_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. It is important here to observe that neither the pair (A, B_0) nor (A, B_1) are controllable, however, the pair $(A, -g'(1))$ is. We choose $X_1(0) = X_2(0) = 1$, $u(x,0) = 1 \forall\, x \in [0,1]$. Finally K is chosen such that the eigenvalues of $A - g'(1)K$ are -1 and -2. The response of the system is shown in Figs. 1–2.

V. Conclusion

In this technical note we present new feedback laws for multi-input LTI systems with distributed inputs. The inputs to the system satisfy diffusion or diffusion with counter-convection PDEs. In addition, we develop observers, for LTI systems in which the sensor dynamics satisfy similar, with the controller design case, PDEs. Our designs are based on novel backstepping-forwarding transformations of the finite-dimensional state of the plant and of the infinite-dimensional actuator or sensor states. Based on these transformations we construct Lyapunov functionals with which we prove exponential stability of the closed-loop system, or convergence of the estimation error in the case of observer design. Finally, our controller design is illustrated by a numerical example.

References

[1] Z. Artstein, "Linear systems with delayed controls: A reduction," *IEEE Trans. Autom. Control*, vol. AC-27, no. 4, pp. 869–879, Aug. 1982.

[2] D. Bresch-Pietri and M. Krstic, "Adaptive trajectory tracking despite unknown input delay and plant parameters," *Automatica*, vol. 45, pp. 2074–2081, 2009.

[3] L. Evans*, Partial Differential Equations*. New York: AMS, 1998.

[4] M. Jankovic, "Forwarding, backstepping, and finite spectrum assignment for time delay systems," *Automatica*, vol. 45, pp. 2–9, 2009.

[5] I. Karafyllis, "Finite-time global stabilization by means of time-varying distributed delay feedback," *Siam J. Control Optim*, vol. 45, no. 1, pp. 320–342, 2006.

[6] M. Krstic and A. Smyshlyaev, "Backstepping boundary control for first-order hyperbolic PDEs and application to systems with actuator and sensor delays," *Syst. Control Lett.*, vol. 57, pp. 750–758, 2008.

[7] M. Krstic*, Delay Compensation for Nonlinear, Adaptive and PDE Systems*. Boston, MA: Birkhauser, 2009.

[8] A. Z. Manitius and A. W. Olbrot, "Finite spectrum assignment for systems with delays," *IEEE Trans. Autom. Control*, vol. AC-24, no. 4, pp. 541–553, Aug. 1979.

[9] F. Mazenc, S. Mondie, and R. Francisco, "Global asymptotic stabilization of feedforward systems with delay at the input," *IEEE Trans. Autom. Control*, vol. 49, no. 5, pp. 844–850, May 2004.

[10] S.-I. Niculescu and A. M. Annaswamy, "An adaptive smith-controller for time-delay systems with relative degree $n \leq 2$," *Syst. Control Lett.*, vol. 49, pp. 347–358, 2003.

[11] J.-P. Richard, "Time-delay systems: An overview of some recent advances and open problems," *Automatica*, vol. 39, pp. 1667–1694, 2003.

[12] O. J. M. Smith, "A controller to overcome dead time," *ISA*, vol. 6, pp. 28–33, 1959.

[13] E. Schuster and M. Krstic, "Control of a nonlinear PDE system arising from non-burning tokamak plasma transport dynamics," *Int. J. Control*, vol. 76, pp. 1116–1124, 2003.

[14] G. A. Susto and M. Krstic, "Control of PDE-ODE cascades with Neumann interconnections," *J. Franklin Inst.*, vol. 347, pp. 284–314, 2010.
[15] X. Zhang and E. Zuazua, "Polynomial decay and control of a 1-d hyperbolic-parabolic coupled system," *J. Differ. Equations*, vol. 204, pp. 380–438, 2004.
[16] Z. Zhong, *Robust Control of Time-Delay Systems*. New York: Springer, 2006.

Sufficient Conditions for Local Asymptotic Stability and Stabilization for Discrete-Time Varying Systems

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***Abstract*—The purpose of this paper is to establish sufficient conditions for local asymptotic stability and feedback stabilization for discrete-time systems with time depended dynamics. Our main results constitute generalizations of those developed by same authors in a recent paper, published in same journal, for the case of continuous-time systems.**

***Index Terms*—Asymptotic stability, averaging, discrete-time systems, stabilization.**

Notations: We adopt the following notations. For $x \in \mathbb{R}^n$, $|x|$ denotes its usual Euclidean norm. Given a matrix $A \in \mathbb{R}^{n\times m}$ we denote by $|A| := \sup_{x\neq 0}(|Ax|/|x|)$ its induced norm. By $S[0,R]$ we denote the closed ball of radius $R>0$ around zero $0\in\mathbb{R}^n$. $\mathcal{N}$ denotes the set of all C^0 functions $\phi:\mathbb{R}^+\to\mathbb{R}^+$ and $\mathcal{K}$ is the set of all functions $\phi\in\mathcal{N}$ which are strictly increasing and vanishing at zero. $\mathcal{K}_\infty$ denotes the subset of $\mathcal{K}$ that constitutes by all $\phi\in\mathcal{K}$ with $\phi(t)\to\infty$ as $t\to\infty$.

I. Introduction

The present work provides sufficient conditions for local asymptotic stability and feedback stabilization for the case of discrete-time systems with time depended dynamics. Our results generalize those in existing works (see for instance [1], [2], [4], [5], [8]). Propositions 1 and 2 in Section II are the main results of the paper establishing Lyapunov-like sufficient conditions for asymptotic stability for systems

$$x(n+1)=f\left(n,x(n)\right),\ (n,x)\in\mathbb{N}\times\mathbb{R}^n. \tag{1}$$

These results constitute, in some sense, the discrete analogue to [9, Proposition 1]. It should be emphasized however, that Proposition 1 and 2, as well as the averaging result of Proposition 5 in Section IV, are based on weaker hypotheses than the discrete-analogue conditions imposed in earlier works concerning continuous-time systems (see for instance, [2], [3], [7], [9] and relative references therein). The result of Proposition 1 is applied in Sections III and IV for the establishment of sufficient conditions for the solvability of the feedback stabilization problem for control systems

$$x(n+1)=F\left(n,x(n),u(n)\right),\ (n,x,u)\in\mathbb{N}\times\mathbb{R}^n\times\mathbb{R}^m \tag{2}$$

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and to derive an averaging type sufficient condition for local asymptotic stability for the case

$$x(n+1)=x(n)+\varepsilon f\left(\varepsilon,n,x(n)\right),\ (\varepsilon,n,x)\in\mathbb{R}\times\mathbb{N}\times\mathbb{R}^n,\ \varepsilon>0. \tag{3}$$

We next provide the concepts of stability, local asymptotic stability and local exponential stability for the case (1). In what follows, we assume that $0\in\mathbb{R}^n$ is an equilibrium, i.e., $f(\cdot,0)=0$. We say that $0\in\mathbb{R}^n$ is stable with respect to (1), if for each $\varepsilon>0$ and given bounded $I\subset\mathbb{N}$ there exists a constant $\delta=\delta(\varepsilon,I)>0$ such that

$$|x(n_0)|\le\delta\Rightarrow|x(n)|\le\varepsilon,\ \forall n\ge n_0, n_0\in I \tag{4}$$

where $x(n)=x(n,n_0,x_0)$, $n=n_0,n_0+1,n_0+2,\ldots$ denotes the solution of (1) initiated from $x_0:=x(n_0)$ at time n_0. We say that $0\in\mathbb{R}^n$ is an attractor for (1), if there exists a constant $\rho>0$ such that for every $\varepsilon>0$ and given bounded $I\subset\mathbb{N}$, a time $\tau=\tau(\varepsilon,I)\in\mathbb{N}$ can be found with

$$|x(n_0)|\le\rho\Rightarrow|x(n)|\le\varepsilon,\ \forall n\ge n_0+\tau,\ n_0\in I. \tag{5}$$

We say that (1) is Asymptotically Stable (AS) (at zero $0\in\mathbb{R}^n$), if zero is stable and an attractor. We say that (1) is Uniformly in time Asymptotically Stable (UAS), if both (4) and (5) hold for every $n_0\in\mathbb{N}$ and for δ and τ depending only on ε. We say that (1) is Exponentially AS (expo-AS), if there exists a constant $\lambda>0$ such that for any given bounded $I\subset\mathbb{N}$, a constant $C=C(I)>0$, $n_0\in I$ can be found with

$$\begin{aligned}|x(n)|\le C\,|x(n_0)|\exp\left(-\lambda(n-n_0)\right),&\\ \forall n\ge n_0, n_0\in I, x(n_0)\,near\ zero.&\end{aligned} \tag{6}$$

Finally, (1) is Exponentially UAS (expo-UAS), if (6) holds for every $n_0\in\mathbb{N}$ and for certain $C>0$ being independent of the initial values n_0 of time.

II. Main Result

The aim of this section is to establish an extension of the main result in [9] for the discrete-time systems (1). We assume that there exists a constant $R>0$ such that the following properties hold:

A1. There exists a function $L\in\mathcal{N}$ such that

$$|f(n,x)|\le L(n)|x|\ \forall(n,x)\in\mathbb{N}\times S[0,R] \tag{7}$$

moreover, we assume that one of the following conditions is fulfilled:

A2. There exist functions $V:\mathbb{N}\times\mathbb{R}^n\to\mathbb{R}^+$, $a,b\in\mathcal{K}_\infty$, $c\in\mathcal{N}$, $r\in\mathcal{K}$, a sequence $\{\sigma_i\ge0, i\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}\}$, a function $m_0\in\mathcal{N}$ and a constant $m>0$, such that

$$\sum_{i=0}^{\infty}\sigma_i=\infty \tag{8a}$$

$$a\left(|x|\right)\le V(n,x)\le b\left(|x|\right)c(n),\ \forall(n,x)\in\mathbb{N}\times S[0,R] \tag{8b}$$

and further the following hold for the solution $x(\cdot)=x(\cdot,\ell_0,x_0)$, $(\ell_0,x_0)\in\mathbb{N}\times S[0,R]$, $x_0=x(\ell_0)$ of (1)

$$\begin{aligned}&V\left(n_{i+1},x(n_{i+1})\right)-V\left(n_i,x(n_i)\right)\le-\sigma_i r\left(V\left(n_i,x(n_i)\right)\right),\\ &n_i=n_i(\ell_0,x_0), x_0\in S[0,R]\ for\ i\in\mathbb{N}_0\ away\ from\ zero,\\ &provided\ that\ x(\nu)\in S[0,R],\\ &\qquad\nu=n_i,n_i+1,n_i+2,\ldots,n_{i+1}\end{aligned} \tag{8c}$$