Lyapunov Stability of Linear Predictor Feedback for Distributed Input Delays

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***Abstract*—Compensation of distributed delays in MIMO, LTI systems is achieved using Artstein's reduction method. In this technical note, we construct a Lyapunov functional for the resulting closed-loop system and establish exponential stability. The key element in our work is the introduction of an infinite-dimensional forwarding-backstepping transformation of the infinite-dimensional actuator states. We illustrate the construction of the Lyapunov functional with a detailed example of a single-input system, in which the input is entering through two individual channels with different delays. Finally, we develop an observer equivalent to the predictor feedback design, for the case of distributed sensor delays and prove exponential convergence of the estimation error.**

***Index Terms*—Delays, distributed parameter systems, linear systems.**

I. Introduction

Delay compensation for linear [1], [2], [8], [10], [11], [15], [16] and nonlinear [5], [6], [12] systems is achieved using predictor-based techniques [4], [13]. In [1] (see also [11]) the following system is considered:

$$\dot{X}(t) = AX(t) + \int_0^D B(\sigma)U(t-\sigma)d\sigma \tag{1}$$

where $X(t) \in \mathbb{R}^n$, $U(t) \in \mathbb{R}^m$ and $D > 0$. Such systems can appear in population dynamics [1], in networked control systems [14], or in liquid mono-propellant rocket motors (see [16] and the references therein). For this system and under the controllability condition of the pair

$$\left(A, \int_0^D e^{-A\sigma}B(\sigma)d\sigma\right) \tag{2}$$

the following controller was developed which achieves asymptotic stability for any $D > 0$:

$$U(t) = Ke^{AD}\left(X(t) + \int_{t-D}^{t}\int_{t-s}^{D} e^{A(t-s-\sigma)}B(\sigma)d\sigma U(s)ds\right) \tag{3}$$

where the control gain K may be designed by a LQR/Riccati approach, pole placement, or some other method that makes $A + \int_0^D e^{-A\sigma}B(\sigma)d\sigma K$ Hurwitz. The above approach (although neither explicitly stated in [1] nor in [11]) works in the case of multi-input systems with different delays in each individual input channel. To see this, one can consider for example the case where $B(\sigma) = [\,B_\delta(\sigma) \quad B_2(\sigma)\,]$, with

$$B_\delta(\sigma) = \left\{\begin{array}{ll} B_1(\sigma), & \text{if } \sigma \le D_1 \\ 0, & \text{if } \sigma > D_1 \end{array}\right\}. \tag{4}$$

Manuscript received April 04, 2010; revised August 13, 2010; accepted October 29, 2010. Date of publication November 15, 2010; date of current version March 09, 2011. Recommended by Associate Editor K. Morris.

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Digital Object Identifier 10.1109/TAC.2010.2092030

In this case system (1) becomes

$$\dot{X}(t) = AX(t) + \sum_{i=1}^{2}\int_0^{D_i} B_i(\sigma)U_i(t-\sigma)d\sigma \tag{5}$$

where $D_2 = D > D_1$.

However, a Lyapunov functional for the closed-loop systems (1) and (3) is not available. The benefits of constructing a Lyapunov functional for predictor feedback controllers was highlighted in [7]. Having a Lyapunov functional available, one can derive an inverse-optimal controller, prove robustness of the predictor feedback to a small mismatch in the actuator delay or study the disturbance attenuation properties of the closed-loop system. In [7], the following single-input system is considered:

$$\dot{X}(t) = AX(t) + BU(t-D) \tag{6}$$

together with the predictor-based controller

$$U(t) = Ke^{AD}X(t) + K\int_{t-D}^{t} e^{A(t-\theta)}BU(\theta)d\theta \tag{7}$$

from [1], [10], [11]. Inverse optimality and robustness to delay mismatch is then proved for the closed-loop systems (6) and (7), using a Lyapunov functional. In addition, a time-varying Lyapunov functional is constructed in [9] for the case of system (6) but when the delay is a function of time i.e., $D = D(t)$. The Lyapunov functionals in [7], [9] are constructed based on the backstepping method for PDEs [8].

Yet, the backstepping method is not applicable neither in the case of single-input systems with distributed input delay nor in the case of multi-input systems with different delays in each individual input channel. This is since the system that is comprised of the finite-dimensional state $X(t)$ and the infinite-dimensional actuator states $U(t + x - D)$, $x \in [0, D]$, are not in the strict-feedback form. In this technical note, a novel forwarding-backstepping transformation of the infinite-dimensional actuator states is introduced to transform the system to an exponentially stable system, whose stability properties can be studied using a quadratic Lyapunov functional. By explicitly finding the inverse transformations, exponential stability of the system in the original variables is established.

We start in Section II with an introduction of the predictor feedback under multiple distributed input delays, and present a transformation of the finite-dimensional state $X(t)$ from [1], [11]. Then, we establish exponential stability of the closed-loop system using our novel infinite-dimensional transformations of the actuator states. In Section III we develop a dual of the predictor-based controller and design an infinite-dimensional observer which compensates the sensor delays. Finally, in Section IV we present an example that is worked out in detail to demonstrate the construction of the Lyapunov functional.

II. Predictor Feedback for Distributed Input Delays

We consider the system

$$\dot{X}(t) = AX(t) + \sum_{i=1}^{2}\int_0^{D_i} B_i(\sigma)U_i(t-\sigma)d\sigma \tag{8}$$

where $X(t) \in \mathbb{R}^n$, $U_1(t), U_2(t) \in \mathbb{R}$ and $D_1, D_2 > 0$. For notational simplicity we consider a two-input case. The same analysis can be carried out for an arbitrary number of inputs with different delays in each individual input channel. For this system, controller

(3) achieves asymptotic stability for any $D_1,\ D_2\ >\ 0$ under the controllability condition of the pair $(A, [\,B_{D_1}\quad B_{D_2}\,])$, where $B_{D_i} = \int_0^{D_i} e^{-A\sigma} B_i(\sigma) d\sigma$, $i = 1, 2$. The predictor feedback (3) can be written as

$$U(t) = \begin{bmatrix} U_1(t) \\ U_2(t) \end{bmatrix} = \begin{bmatrix} K_1 e^{AD_1} \\ K_2 e^{AD_2} \end{bmatrix} Z(t) \tag{9}$$

$$Z(t) = X(t) + \sum_{i=1}^{2} \int_0^{D_i} \int_{D_i - y}^{D_i} e^{A(D_i - y - \sigma)} B_i(\sigma) d\sigma \times U_i(t + y - D_i) dy, \quad i = 1, 2 \tag{10}$$

transformation (10) can be found in [1] and [11]. We are now ready to state the main result of this section.

Theorem 1: Consider the closed-loop systems consisting of the plant (8) and the controller (9). Let the pair $(A, [\,B_{D_1}\quad B_{D_2}\,])$ be completely controllable and choose K_1 and K_2 such that $A + B_{D_1} K_1 e^{AD_1} + B_{D_2} K_2 e^{AD_2}$ is Hurwitz. Then for any initial conditions $u_i(\cdot, 0) \in L^2(0, D_i)$, $i = 1, 2$ the closed-loop system has a unique solution $(X(t), u_1(\cdot, t), u_2(\cdot, t)) \in C([0, \infty), \mathbb{R}^n \times L^2(0, D_1) \times L^2(0, D_2))$ which is exponentially stable in the sense that there exist positive constants μ and ρ, such that

$$\Omega(t) \leq \mu \Omega(0) e^{-\rho t} \tag{11}$$

$$\Omega(t) = |X(t)|^2 + \sum_{i=1}^{2} \int_0^{D_i} U_i^2(t - \theta) d\theta. \tag{12}$$

Moreover, if the initial conditions $u_i(\cdot, 0)$, $i = 1, 2$ are compatible with the control law (9) and belong to $H^1(0, D_i)$, $i = 1, 2$, then $(X(t), u_1(\cdot, t), u_2(\cdot, t)) \in C^1([0, \infty), \mathbb{R}^n \times L^2(0, D_1) \times L^2(0, D_2)) \cap C([0, \infty), \mathbb{R}^n \times H^1(0, D_1) \times H^1(0, D_2))$ is the classical solution of the closed-loop system.

Remark 2.1: Before we start the proof of Theorem 1 we give some insight into the main challenges that one faces by considering the problem of constructing a Lyapunov functional for system (5) instead of system (6) that was considered in [7]. System (6) is an ODE-PDE cascade in the strict-feedback form. To see this we re-write (6) as $\dot{X}(t) = AX(t) + Bu(0, t)$, where $u(x, t)$ satisfies $\partial_t u(x, t) = \partial_x u(x, t)$ and $u(D, t) = U(t)$. One can now observe that the finite-dimensional state $X(t)$ and the infinite-dimensional state $u(x, t), x \in [0, D]$ are in the strict-feedback form since $u(x, t)$ affects the $X(t)$ block only through $x = 0$. On the other hand in (5) it is clear that $u(x, t)$ affects the $X(t)$ block through all of the spatial domain $x \in [0, D]$ and hence the overall system is not in the strict-feedback form. For this reason, a spatially casual backstepping transformation of the infinite-dimensional actuator state which employs a Volterra integral, as the one in [7] and [8], does not apply.

Proof: We first rewrite the plant (8) as

$$\dot{X}(t) = AX(t) + \sum_{i=1}^{2} \int_0^{D_i} B_i(D_i - y) u_i(y, t) dy \tag{13}$$

$$\partial_t u_1(x, t) = \partial_x u_1(x, t) \tag{14}$$

$$u_1(D_1, t) = U_1(t) \tag{15}$$

$$\partial_t u_2(z, t) = \partial_z u_2(z, t) \tag{16}$$

$$u_2(D_2, t) = U_2(t) \tag{17}$$

where $x \in [0, D_1]$ and $z \in [0, D_2]$. Note that $u_1(x, t) = U_1(t + x - D_1)$ and $u_2(z, t) = U_2(t + z - D_2)$. Note also that the transformation (10) can be written as

$$Z(t) = X(t) + \sum_{i=1}^{2} \int_0^{D_i} \int_{D_i - y}^{D_i} e^{A(D_i - y - \sigma)} B_i(\sigma) d\sigma u_i(y, t) dy. \tag{18}$$

Consider the transformations of the infinite-dimensional actuator states $u_1(x, t)$ and $u_2(z, t)$

$$w_1(x, t) = u_1(x, t) - \gamma_1(x) X(t) - \int_0^x p_1(x, y) u_1(y, t) dy + \int_x^{D_1} k_1(x, y) u_1(y, t) dy - \int_0^{D_2} q_1(x, y) u_2(y, t) dy \tag{19}$$

$$w_2(z, t) = u_2(z, t) - \gamma_2(z) X(t) - \int_0^z p_2(z, y) u_2(y, t) dy + \int_z^{D_2} k_2(z, y) u_2(y, t) dy - \int_0^{D_1} q_2(z, y) u_1(y, t) dy \tag{20}$$

where the kernels $\gamma_i(\cdot)$, $p_i(\cdot)$, $k_i(\cdot)$ and $q_i(\cdot)$ for $i = 1, 2$ are to be specified. Using transformations (18)–(20) and relations (9)–(10) we transform the plant (13)–(17) into the "target system"

$$\dot{Z}(t) = \left(A + B_{D_1} K_1 e^{AD_1} + B_{D_2} K_2 e^{AD_2} \right) Z(t) \tag{21}$$

$$\partial_t w_1(x, t) = \partial_x w_1(x, t) - q_1(x, D_2) K_2 e^{AD_2} Z(t) \tag{22}$$

$$w_1(D_1, t) = 0 \tag{23}$$

$$\partial_t w_2(z, t) = \partial_z w_2(z, t) - q_2(z, D_1) K_1 e^{AD_1} Z(t) \tag{24}$$

$$w_2(D_2, t) = 0. \tag{25}$$

We first prove (21). Differentiating with respect to time (18) and using (13)–(14), (16) we obtain $\dot{Z}(t) = AX(t) + \sum_{i=1}^{2} \left(\int_0^{D_i} B_i(D_i - y) u_i(y, t) dy + \int_0^{D_i} \int_{D_i - y}^{D_i} e^{A(D_i - y - \sigma)} B_i(\sigma) d\sigma \partial_y u_i(y, t) dy \right)$. Using integration by parts together with (9), (15), (17) and Leibniz's differentiation rule, we obtain (21). We next prove (22)–(23) (the proof of (24)–(25) follows exactly the same pattern). To obtain (22) we differentiate (19) with respect to t and x. Using relations (14), (16) and integration by parts, after subtracting the resulting expressions for the time and spatial derivatives of (19) in order to get (22), we obtain a system of ODEs and PDEs which is well posed and can be solved explicitly to give

$$\gamma_1(x) = K_1 e^{Ax} \tag{26}$$

$$p_1(x,y) = \int_{D_1-y}^{D_1} K_1 e^{A(D_1+x-y-\sigma)} B_1(\sigma) d\sigma \tag{27}$$

$$k_1(x,y) = \int_{0}^{D_1-y} K_1 e^{A(D_1+x-y-\sigma)} B_1(\sigma) d\sigma \tag{28}$$

$$q_1(x,y) = \int_{D_2-y}^{D_2} K_1 e^{A(D_2+x-y-\sigma)} B_2(\sigma) d\sigma. \tag{29}$$

Using (9) we get (22)–(23). Similarly, one obtains the kernels of (20) which can be derived from (26)–(29) by changing the index 1 into 2 and the spatial variable x into z. The next step is deriving the inverse transformations of (19)–(20). We postulate the inverse transformation of (19) in the form

$$u_1(x,t) = w_1(x,t) + \delta_1(x) Z(t) - \int_x^{D_1} m_1(x,y) w_1(y,t) dy. \tag{30}$$

Taking the time and the spatial derivatives of the above transformation, using integrations by parts together with (21)–(23) and the fact that $q_1(x,D_2) = K_1 e^{Ax} B_{D_2}$, after subtracting the resulting expressions for the time and spatial derivatives, we conclude that relations (14)–(15) hold (by taking into account also (9)) if $\delta_1(x) = K_1 e^{AD_1} e^{A_{cl}(x-D_1)} + K_1 e^{Ay} B_{D_2}\, K_2 e^{AD_2} e^{A_{cl}(x-y)} dy - \int_x^{D_1} \int_y^{D_1} g_1(y-r) K_1\, e^{Ar} B_{D_2} K_2 e^{AD_2} dr e^{A_{cl}(x-y)} dy$, $A_{cl} = A + B_{D_1} K_1 e^{AD_1} + B_{D_2} K_2 e^{AD_2}$, $m_1(x,y) = g_1(x-y)$. To establish that (30) is indeed the inverse transformation of (19) one has to uniquely determine $g_1(x-y)$. We find next an explicit expression for $g_1(x-y)$. To do so, we substitute $Z(t)$ and $w_1(x,t)$ from (18) and (19) respectively, into (30). Matching the terms for $X(t)$, $u_1(y,t)$ and $u_2(y,t)$ and after some algebraic manipulations we conclude that $g_1(x-y)$ must satisfy for all $x \in [0, D_1]$

$$0 = -K_1 e^{Ax} + \delta_1(x) + \int_x^{D_1} g_1(x-y) K_1 e^{Ay} dy \tag{31}$$

$$\begin{aligned}
0 = {} & K_1 e^{Ax} \int_x^{D_1} \int_{D_1-y}^{D_1} e^{A(D_1-y-\sigma)} B_1(\sigma) d\sigma u_1(y,t) dy \\
& + K_1 e^{Ax} \int_x^{D_1} \int_0^{D_1-y} \\
& \times e^{A(D_1-y-\sigma)} B_1(\sigma) d\sigma u_1(y,t) dy \\
& - \int_x^{D_1} g_1(x-y) u_1(y,t) dy \\
& + \int_0^{D_1} \left(-K_1 e^{Ax} + \delta_1(x) + \int_x^{D_1} g_1(x-r) K_1 e^{Ar} dr \right) \\
& \times \int_{D_1-y}^{D_1} e^{A(D_1-y-\sigma)} B_1(\sigma) d\sigma \\
& \times u_1(y,t) dy - \int_x^{D_1} g_1(x-y) \\
& \times \int_y^{D_1} K_1 e^{Ay} e^{A(D_1-r)} B_{D_1} u_1(r,t) dr dy.
\end{aligned} \tag{32}$$

We can now substitute the expression for $\delta_1(x)$ into (31)–(32) and then find $g_1(x-y)$ that satisfies conditions (31)–(32). Instead, assuming for the moment that $g_1(x-y)$ and $\delta_1(x)$ satisfy (31) we remain with the following condition for all $x, y \in [0, D_1]$

$$0 = K_1 e^{A(x-y)} B_{1e} - g_1(x-y) - \int_x^y g_1(x-r) K_1 e^{A(r-y)} dr B_{1e} \tag{33}$$

where $B_{1e} = e^{AD_1} B_{D_1}$. Assuming a form for $g_1(x-y)$ as $g_1(x-y) = K_1 R_1(x-y) B_{1e}$, we conclude that (33) is satisfied if $R_1(0) = I$ together with $R_1'(w) = R_1(w)(A + B_{1e} K_1)$. Therefore

$$g_1(x) = K_1 e^{(A+B_{1e}K_1)x} B_{1e}. \tag{34}$$

Using (34) we can simplify $\delta_1(x)$. After some algebra we rewrite $\delta_1(x)$ as

$$\delta_1(x) = K_1 e^{(A+B_{1e}K_1)(x-D_1)} e^{AD_1}. \tag{35}$$

Finally, after some algebra one can verify that indeed relations (34)–(35) satisfy condition (31). Similarly, the inverse transformation of (20) is given by

$$u_2(z,t) = w_2(z,t) + \delta_2(z) Z(t) - \int_z^{D_2} m_2(z,y) w_2(y,t) dy \tag{36}$$

and $B_{2e} = e^{AD_2} B_{D_2}$, $m_2(z,y) = g_2(z-y) = K_2 e^{(A+B_{2e}K_2)(z-y)} B_{2e}$, $\delta_2(z) = K_2 e^{(A+B_{2e}K_2)(z-D_2)} e^{AD_2}$ Using (18) together with (30) and (36) we get $X(t)$ as a function of $w_1(y,t)$ and $w_2(y,t)$. We have

$$\begin{aligned}
X(t) = {} & \left(I - \sum_{i=1}^{2} \int_0^{D_i} \int_{D_i-y}^{D_i} e^{A(D_i-y-\sigma)} B_i(\sigma) d\sigma \times \delta_i(y) dy \right) Z(t) - \sum_{i=1}^{2} \int_0^{D_i} \int_{D_i-y}^{D_i} \\
& \times e^{A(D_i-y-\sigma)} B_i(\sigma) d\sigma \left(w_i(y,t) - \int_y^{D_i} m_i(y,r) w_i(r,t) dr \right) dy.
\end{aligned} \tag{37}$$

Consider now the Lyapunov functional for the target system (21)–(25)

$$V(t) = Z(t)^T P Z(t) + \alpha \int_0^{D_1} (1+x) w_1^2(x,t) dx + \beta \int_0^{D_2} (1+z) w_2^2(z,t) dz \tag{38}$$

where the positive parameters α, β are to be chosen later and $P = P^T > 0$ is the solution to the following Lyapunov equation:

$$\left(A + B_{D_1} K_1 e^{AD_1} + B_{D_2} K_2 e^{AD_2} \right)^T P + P \left(A + B_{D_1} K_1 e^{AD_1} + B_{D_2} K_2 e^{AD_2} \right) = -Q \tag{39}$$

for some $Q = Q^T > 0$, K_1, and K_2. For the time derivative of $V(t)$ along (21)–(25) we obtain $\dot{V}(t) \leq -\lambda_{\min}(Q)|Z(t)|^2 - \alpha\, w_1^2(0,t) - \alpha \int_0^{D_1} w_1^2(x,t) dx - \int_0^{D_1} 2\,(\sqrt{\alpha}/\sqrt{2}) w_1(x,t)(1+x) K_1\, e^{Ax} B_{D_2} K_2 e^{AD_2} \sqrt{\alpha}\sqrt{2} dx Z(t) - \beta w_2^2(0,t) - \beta \int_0^{D_2} w_2^2(z,t) dz - \int_0^{D_2} 2(\sqrt{\beta}/\sqrt{2}) w_2(z,t)(1+z) K_2 e^{Az}\, B_{D_1} K_1 e^{AD_1} \sqrt{\beta}\sqrt{2} dz Z(t)$, where $\lambda_{\min}(Q)$ is the smallest eigenvalue of Q in (39). Applying Young's inequality we get $\dot{V}(t) \leq -(\lambda_{\min}(Q) - \alpha\gamma - \beta\delta)$

$|Z(t)|^2 - (\alpha/2)\int_0^{D_1} w_1^2(x,t)\, dx - (\beta/2)\int_0^{D_2} w_2^2(z,t)dz$, where $\gamma = 2D_1\left|(1+D_1)K_1e^{AD_1}B_{D_2}K_2e^{AD_2}\right|^2$ and $\delta = 2D_2\left|(1+D_2)K_2e^{AD_2}B_{D_1}K_1e^{AD_1}\right|^2$. Hence by choosing $\alpha = \lambda_{\min}(Q)/4\gamma$ and $\beta = \lambda_{\min}(Q)/4\delta$, and by setting $\rho = \min\{\lambda_{\min}(Q)/2\lambda_{\max}(P), \alpha/2(1+D_1), \beta/2(1+D_2)\}$, where $\lambda_{\max}(P)$ is the largest eigenvalue of P in (39) we get $\dot{V}(t) \le -\rho V(t)$. Using the comparison principle we get $V(t) \le V(0)e^{-\rho t}$. To prove stability in the original variables $X(t)$ and $u_1(x,t), u_2(z,t)$ it is sufficient to show that

$$\underline{M}\Omega(t) \le V(t) \le \overline{M}\Omega(t) \tag{40}$$

for some positive $\overline{M}$ and $\underline{M}$, since then $\Omega(t) \le (\overline{M}/\underline{M})\Omega(0)e^{-\rho t}$, and Theorem 1 is proved with $\mu = \overline{M}/\underline{M}$. Using (18), and applying Young and Cauchy-Schwartz's inequalities we get $|Z(t)|^2 \le m(|X(t)|^2 + \int_0^{D_1} u_1(x,t)^2dx + \int_0^{D_2} u_2(z,t)^2dz)$, where $m = 3\max\{1, \max_{i=1,2}\{(\int_{D_i-y}^{D_i} e^{A(D_i-y-\sigma)}B_i(\sigma)d\sigma)^2\}\}$. Moreover, using (19)–(20) together with Young and Cauchy-Schwartz's inequalities we obtain $\int_0^{D_i} w_i(y,t)^2dy \le m_i(|X(t)|^2 + \int_0^{D_1} u_1(x,t)^2dx + \int_0^{D_2} u_2(z,t)^2dz)$, $i = 1, 2$, where $m_i = 4D_i\max\{1, f_{1i}, f_{2i}, f_{3i}\}$, $f_{1i} = \sup_{y\in[0,D_i]}\gamma_i(y)^2$, $f_{2i} = D_i\sup_{0\le y\le x\le D_i} p_i(x,y)^2 + D_i\sup_{0\le x\le y\le D_i} k_i(x,y)^2$, $f_{3i} = D_j\sup_{x\in[0,D_i],y\in[0,D_j]} q_i(x,y)^2$ and $i, j = 1, 2, j \ne i$. With $\overline{M} = \max\{\lambda_{\max}(P), \alpha(1+D_1), \beta(1+D_2)\}\max\{m, m_1, m_2\}$ the upper bound in (40) is proved. Similarly using (30), (36), (37) we get $\underline{M} = \min\{\lambda_{\min}(P), \alpha, \beta\}/\max\{b, b_1, b_2\}$, where $b = \max\{g_1, g_2\}$ and $g_1 = 9\left(1 + \sum_{i=1}^2 D_i\sup_{y\in[0,D_i]}\left(\int_{D_i-y}^{D_i} e^{A(D_i-y-\sigma)}B_i(\sigma)d\sigma\right)^2\right)$, $g_2 = \max_{i=1,2}\{6D_i\left(1 + D_i\sup_{0\le x\le y\le D_i} m_i(x,y)^2\right)l\}$ and $l = \sup_{y\in[0,D_i]}\left(\int_{D_i-y}^{D_i} e^{A(D_i-y-\sigma)}B_i(\sigma)d\sigma\right)^2$. Finally, $b_i = 3\max\{v_{1i}, v_{2i}\}$, $i = 1, 2$, with $v_{1i} = 1 + D_i^2\sup_{0\le x\le y\le D_i} m_i(x,y)^2$ and $v_{2i}D_i\sup_{0\le x\le D_i}\delta_i(x)^2$. Moreover, from (21) we conclude that $Z(t)$ is bounded and converges exponentially to zero. From (19)–(20) one can conclude that $w_i(\cdot,0) \in L^2(0,D_i)$, $i = 1, 2$ and thus it follows from (22)–(25) that $w_i(\cdot,t) \in C([0,\infty), L^2(0,D_i))$, $i = 1, 2$. Using the inverse transformations (30) and (36) we can conclude that $u_i(\cdot,t) \in C([0,\infty), L^2(0,D_i))$, $i = 1, 2$. The uniqueness of weak solution is proved using the uniqueness of weak solution to the boundary problems (22)–(25) (see, e.g., [3]). Similarly, when $u_i(\cdot,0)$, $i = 1, 2$ are compatible with the control law (9) and belong to $H^1(0,D_i)$, $i = 1, 2$, from (19)–(20) we have that $w_i(\cdot,0) \in H^1(0,D_i)$, $i = 1, 2$ and therefore using (22)–(25) that $w_i(\cdot,t) \in C^1([0,\infty), L^2(0,D_i)) \cap C([0,\infty), H^1(0,D_i))$, $i = 1, 2$. Using relations (30), (36) with [3] (section 2.1 and section 2.3) the theorem is proved. ■

III. Observer Design With Distributed Sensor Delays

We consider the system

$$\dot{X}(t) = AX(t) + BU(t) \tag{41}$$

$$Y_1(t) = \int_0^{D_1} C_1(\sigma)X(t-\sigma)d\sigma \tag{42}$$

$$Y_2(t) = \int_0^{D_2} C_2(\sigma)X(t-\sigma)d\sigma \tag{43}$$

which can be written equivalently as $\dot{X}(t) = AX(t) + BU(t)$, $\partial_t\xi_1(x,t) = \partial_x\xi_1(x,t) + C_1(x)X(t)$, $\xi_1(D_1,t) = 0$, $\partial_t\xi_2(z,t) = \partial_z\xi_2(z,t) + C_2(z)X(t)$, $\xi_2(D_2,t) = 0$, $Y_1(t) = \xi_1(0,t)$ and $Y_2(t) = \xi_2(0,t)$. Next we state a new observer that compensates the sensor delays and prove exponential convergence of the resulting observer error system.

Theorem 2: Consider the system (41)–(43) and let the pair $\left(A, \begin{bmatrix} C_{D_1} \\ C_{D_2} \end{bmatrix}\right)$ be observable, where

$$C_{D_i} = \int_0^{D_i} C_i(\sigma)e^{-A\sigma}d\sigma, \quad i = 1, 2. \tag{44}$$

Define the observer

$$\dot{\hat{X}}(t) = A\hat{X}(t) + BU(t) + \sum_{i=1}^2 L_i\left(Y_i(t) - \hat{Y}_i(t)\right) \tag{45}$$

$$\partial_t\hat{\xi}_1(x,t) = \partial_x\hat{\xi}_1(x,t) + C_1(x)\hat{X}(t) + \gamma_1(x)\sum_{i=1}^2 L_i\left(Y_i(t) - \hat{Y}_i(t)\right) \tag{46}$$

$$\hat{\xi}_1(D_1,t) = 0 \tag{47}$$

$$\partial_t\hat{\xi}_2(z,t) = \partial_z\hat{\xi}_2(z,t) + C_2(z)\hat{X}(t) + \gamma_2(z)\sum_{i=1}^2 L_i\left(Y_i(t) - \hat{Y}_i(t)\right) \tag{48}$$

$$\hat{\xi}_2(D_2,t) = 0 \tag{49}$$

$$\hat{Y}_1(t) = \hat{\xi}_1(0,t) \tag{50}$$

$$\hat{Y}_2(t) = \hat{\xi}_2(0,t) \tag{51}$$

$$\gamma_1(x) = \left(C_{D_1} - \int_0^x C_1(y)e^{-Ay}dy\right)e^{Ax} \tag{52}$$

$$\gamma_2(z) = \left(C_{D_2} - \int_0^z C_2(y)e^{-Ay}dy\right)e^{Az} \tag{53}$$

where L_1 and L_2 are chosen such that the matrix $A - L_1C_{D_1} - L_2C_{D_2}$ is Hurwitz. Then for any $(\xi_i(\cdot,0), \hat{\xi}_i(\cdot,0)) \in L^2(0,D_i)$, $i = 1, 2$ the observer error system has a unique solution $\left(X(t) - \hat{X}(t), \xi_1(\cdot,t) - \hat{\xi}_1(\cdot,t), \xi_2(\cdot,t) - \hat{\xi}_2(\cdot,t)\right) \in C([0,\infty), \mathbb{R}^n \times L^2(0,D_1) \times L^2(0,D_2))$ which is exponentially stable in the sense that there exist positive constants κ and λ such that

$$\Xi(t) \le \kappa\Xi(0)e^{-\lambda t} \tag{54}$$

$$\Xi(t) = |X(t) - \hat{X}(t)|^2 + \int_0^{D_1}(\xi_1(x,t) - \hat{\xi}_1(x,t))^2dx + \int_0^{D_2}(\xi_2(z,t) - \hat{\xi}_2(z,t))^2dz. \tag{55}$$

Proof: Introducing the error variables $\tilde{X}(t) = X(t) - \hat{X}(t)$, $\tilde{\xi}_1(x,t) = \xi_1(x,t) - \hat{\xi}_1(x,t)$ and $\tilde{\xi}_2(z,t) = \xi_2(z,t) - \hat{\xi}_2(z,t)$, we obtain $\dot{\tilde{X}}(t) = A\tilde{X}(t) - L_1\tilde{\xi}_1(0,t) - L_2\tilde{\xi}(0,t)$, $\partial_t\tilde{\xi}_1(x,t) = \partial_x\tilde{\xi}_1(x,t) + C_1(x)\tilde{X}(t) - \gamma_1(x)L_1\tilde{\xi}_1(0,t) - \gamma_1(x)L_2\tilde{\xi}_2(0,t)$, $\tilde{\xi}_1(D_1,t) = 0$, $\partial_t\tilde{\xi}_1(z,t) = \partial_z\tilde{\xi}_2(z,t) + C_2(z)\tilde{X}(t) - \gamma_2(z)L_1\tilde{\xi}_1(0,t) - \gamma_2(z)L_2\tilde{\xi}_2(0,t)$, $\tilde{\xi}_2(D_2,t) = 0$. With the transformations

$$\tilde{\zeta}_1(x,t) = \tilde{\xi}_1(x,t) - \gamma_1(x)\tilde{X}(t) \tag{56}$$

$$\tilde{\zeta}_2(z,t) = \tilde{\xi}_2(z,t) - \gamma_2(z)\tilde{X}(t) \tag{57}$$

and by noting that $\gamma_1(x)$ and $\gamma_2(z)$ in (52)–(53) are the solutions of the boundary value problems $\gamma_1'(x) = \gamma_1(x)A - C_1(x)$ together with $\gamma_1(D_1) = 0$ and $\gamma_2'(z) = \gamma_2(z)A - C_2(z)$ together with $\gamma_2(D_2) = 0$, we get

$$\dot{\tilde{X}}(t) = (A - L_1C_{D_1} - L_2C_{D_2})\tilde{X}(t) - L_1\tilde{\zeta}_1(0,t) - L_2\tilde{\zeta}_2(0,t) \tag{58}$$

$$\partial_t\tilde{\zeta}_1(x,t) = \partial_x\tilde{\zeta}_1(x,t) \tag{59}$$

$$\tilde{\zeta}_1(D_1,t) = 0 \tag{60}$$

$$\partial_t \tilde{\zeta}_2(z,t) = \partial_z \tilde{\zeta}_2(z,t) \tag{61}$$
$$\tilde{\zeta}_2(D_2,t) = 0. \tag{62}$$

To establish exponential stability of the error system we use the following Lyapunov functional

$$V(t) = \tilde{X}^T(t)P\tilde{X}(t) + \alpha \int_0^{D_1} (1+x)\tilde{\zeta}_1^2(x,t)dx + \beta \int_0^{D_2} (1+z)\tilde{\zeta}_2^2(z,t)dz \tag{63}$$

where the positive parameters α, β are to be chosen later and $P = P^T > 0$ satisfies the Lyapunov equation $(A - L_1C_{D_1} - L_2C_{D_2})^T P + P(A - L_1C_{D_1} - L_2C_{D_2}) = -Q$, for some $Q = Q^T > 0$, L_1 and L_2. Taking the time derivative of $V(t)$, using integration by parts in the integrals in (63) and relations (58)–(62) we get $\dot{V}(t) \le -\lambda_{\min}(Q)|\tilde{X}(t)|^2 - \alpha \int_0^{D_1} \tilde{\zeta}_1^2(x,t)dx - \alpha \zeta_1^2(0,t) - \beta \int_0^{D_2} \tilde{\zeta}_2^2(z,t)\, dz - \beta\zeta_2^2(0,t) + (\sqrt{\lambda_{\min}(Q)}|\tilde{X}(t)|/\sqrt{2})(2\sqrt{2}|P||L_1|/\sqrt{\lambda_{\min}(Q)})\ \ |\zeta_1(0,t)| + (\sqrt{\lambda_{\min}(Q)}|\tilde{X}(t)|/\sqrt{2})(2\sqrt{2}|P||L_2|/\sqrt{\lambda_{\min}(Q)})|\zeta_2(0,t)|$. Applying Young's inequality we get $\dot{V}(t) \le -(\lambda_{\min}(Q)/2)|\tilde{X}(t)|^2 - \alpha \int_0^{D_1} \tilde{\zeta}_1^2(x,t)dx - \beta \int_0^{D_2} \tilde{\zeta}_2^2(z,t)dz \le -\gamma V(t)$, where $\alpha = 4|P|^2|L_1|^2/\lambda_{\min}(Q)$, $\beta = 4|P|^2|L_2|^2/\lambda_{\min}(Q)$ and $\gamma = \min\{\lambda_{\min}(Q)/2\lambda_{\max}(P), \alpha/1+D_1, \beta/1+D_2\}$. Similarly to the proof of Theorem 1, in order to establish exponential stability of the observer estimation error it is sufficient to show that $\underline{M}\Xi(t) \le V(t) \le \overline{M}\Xi(t)$ for some positive $\overline{M}$ and $\underline{M}$. Using (56)–(57) with Young's inequality we get $\overline{M} = \max\left\{2, 1 + 2\sum_{i=1}^2 D_i \sup_{y\in[0,D_i]} \gamma_i(y)^2\right\} \times \max\{\lambda_{\max}(P), \alpha(1+D_1), \beta(1+D_2)\}$ and $\underline{M} = \min\{\lambda_{\min}(P), \alpha, \beta\}/\max\left\{2, 1 + 2\sum_{i=1}^2 D_i \sup_{y\in[0,D_i]} \gamma_i(y)^2\right\}$. From (56)–(57) one can conclude that $\tilde{\zeta}_i(\cdot,0) \in L^2(0,D_i)$, $i = 1, 2$ and thus it follows from (59)–(62) that $\tilde{\zeta}_i(\cdot,t) \in C(L^2(0,D_i))$, $i = 1, 2$. Thus from (58) it follows that $\tilde{X}(t)$ is bounded. Using the inverse transformations of (56)–(57) we can conclude that $\tilde{\xi}_i(\cdot,t) \in C(L^2(0,D_i))$, $i = 1, 2$. The uniqueness of weak solution is proved using the uniqueness of weak solution to the boundary problems (59)–(62) (see, e.g., [3]). ■

IV. Example

We consider the case where $B_2(\sigma) = 0, \forall\, \sigma \in [0, D_2]$ and $B_1(\sigma) = B\delta(\sigma) + B_0\delta(D_1 - \sigma)$, where $\delta(\sigma)$ is the Dirac function. In this case system (13)–(17) can be represented as

$$\dot{X}(t) = AX(t) + B_0u(0,t) + Bu(D,t) \tag{64}$$
$$\partial_t u(x,t) = \partial_x u(x,t) \tag{65}$$
$$u(D,t) = U(t) \tag{66}$$

with $u(x,t) = u_1(x,t)$ and $D = D_1$. The transformation of the finite-dimensional state $X(t)$ in relation (18) becomes

$$Z(t) = X(t) + \int_0^D e^{-Ay}B_0u(y,t)dy. \tag{67}$$

Moreover, the infinite-dimensional transformation of the actuator state $u(x,t)$ in (19) together with the kernels (27)–(29) give ($K = K_1$)

$$\begin{aligned} w(x,t) = {} & u(x,t) - Ke^{Ax}X(t) \\ & - K\int_0^x e^{A(x-y)}B_0u(y,t)dy \\ & + K\int_x^D e^{A(D+x-y)}Bu(y,t)dy.. \end{aligned} \tag{68}$$

Using (67) and (68) and since the system has a single input (that is, $q_1(x,y)$ in (22) is zero since it is given by (29)) we get $\dot{Z}(t) = AZ(t) + \left(e^{-AD}B_0 + B\right)u(D,t)$ and $\partial_t w(x,t) = \partial_x w(x,t)$. With

$$\begin{aligned} u(D,t) &= Ke^{AD}Z(t) \\ &= Ke^{AD}X(t) \\ &\quad + K\int_0^D e^{A(D-y)}B_0u(y,t)dy \end{aligned} \tag{69}$$

we get

$$\dot{Z}(t) = \left(A + \left(e^{-AD}B_0 + B\right)Ke^{AD}\right)Z(t) \tag{70}$$
$$\partial_t w(x,t) = \partial_x w(x,t) \tag{71}$$
$$w(D,t) = 0. \tag{72}$$

Relation (30) together with the facts $e^{(A+B_eK)(x-y)} = e^{e^{AD}Acl(x-y)e^{-AD}} = e^{AD}e^{Acl(x-y)}e^{-AD}$ and $B_e = e^{AD}(e^{-AD}B_0 + B)$, where $A_{cl} = A + \left(e^{-AD}B_0 + B\right)Ke^{AD}$, yield the inverse transformation of (68) as

$$u(x,t) = w(x,t) + K_e e^{A_{cl}x}Z(t) - K_e\int_x^D e^{A_{cl}(D+x-y)} \times \left(e^{-AD}B_0 + B\right)w(y,t)dy \tag{73}$$

with $K_e = Ke^{AD}e^{-A_{cl}D}$. Finally, we express $X(t)$ in terms of $Z(t)$ and $w(x,t)$ as

$$\begin{aligned} X(t) = {} & \left(I - \int_0^D e^{-Ay}B_0Ke^{AD}e^{Acl(y-D)}dy\right)Z(t) \\ & - \int_0^D e^{-Ay}B_0w(y,t)dy + \int_0^D e^{-Ay}B_0Ke^{AD}e^{-AclD} \\ & \times \int_y^D e^{A_{cl}(D+y-r)}\left(e^{-AD}B_0 + B\right)w(r,t)drdy. \end{aligned} \tag{74}$$

Now one can use the Lyapunov functional (38) to establish exponential stability of the closed-loop system (64)–(66), (69). Instead, we will use the following Lyapunov functional

$$V(t) = X^T(t)PX(t) + \alpha\int_0^D (1+x)w^2(x,t)dx. \tag{75}$$

where α is an arbitrary positive constant. Observe here that the above Lyapunov functional depends on $X(t)$ directly and through $w(x,t)$, while the Lyapunov functional defined in (38) depends on $X(t)$ through $Z(t)$ and $w(x,t)$. Hence, in order to prove exponential stability of the closed-loop system (64)–(66), (69), using (75), we have to derive a relation for the (X, w) cascade.

To see this one has to solve (74) for $Z(t)$ (assuming that the matrix that multiplies $Z(t)$ is invertible) and then plugging the resulting expression into (73) and (69). Then, having on the right hand side of (73) and (69) only $X(t)$ and $w(x,t)$ one can find $u(0,t)$ and $u(D,t)$ only as a function of $X(t)$ and $w(x,t)$. Plugging into (64) $u(0,t)$ and $u(D,t)$ we get the (X, w) cascade. One can show using (75) that this cascade is exponentially stable.

We prove here this result for a simpler case. We set $B_0 = B$. Then (64)–(66) become $\dot{X}(t) = AX(t) + B(u(0,t) + u(D,t))$, $\partial_t u(x,t) = \partial_x u(x,t)$, $u(D,t) = U(t)$. Consequently, by adding (69) and (73) for $x = 0$ we have $u(0,t) + u(D,t) = w(0,t) - Ke^{AD}\int_0^D e^{-A_{cl}y}(e^{-AD} + I)\,Bw(y,t)dy + Ke^{AD}\left(I + e^{-A_{cl}D}\right)Z(t)$. Multiplying (74) on the left with $(I + e^{-AD})$, using the facts that $d\left(e^{-Ay}e^{A_{cl}(y-D)}\right)/dy = e^{-Ay}(I + e^{-AD})BKe^{AD}e^{A_{cl}(y-D)}$, $B_0 = B$ and changing the order of integration in the last integral in (74) we have $(I + e^{-AD})X(t) = \left(I + e^{-AclD}\right)Z(t) - \int_0^D e^{-A_{cl}y}\left(I + e^{-AD}\right)Bw(y,t)dy$. Since

A_{cl} is Hurwitz, the matrix $I + e^{-AclD}$ that multiplies $Z(t)$ in the previous relation is invertible. Hence, by solving the previous relation for $Z(t)$ we get $u(0,t)+u(D,t) = w(0,t)+Ke^{AD}\left(I+e^{-AD}\right)X(t)$. Hence

$$\dot{X}(t) = (A + BK(I+e^{AD}))X(t) + Bw(0,t) \quad (76)$$
$$\partial_t w(x,t) = \partial_x w(x,t) \quad (77)$$
$$w(D,t) = 0. \quad (78)$$

One can now show exponential stability of the closed-loop system (76)–(78) using (75). Exponential stability in the original variables can be then proved by following a similar pattern of calculations as in Section II. In the present case (75) has the form $V(t) = X^T(t)PX(t) + \alpha\int_{t-D}^{t}(1+\theta+D-t)W^2(\theta)d\theta$, $W(\theta) = U(\theta) - Ke^{A(\theta+D-t)}X(t) - K\int_{t-D}^{\theta} e^{A(\theta-\sigma)}BU(\sigma)d\sigma + K\int_{\theta}^{t} e^{A(D+\theta-\sigma)}BU(\sigma)d\sigma$ where $t-D \le \theta \le t$. Observe here that in order to choose K in (76) such that $A + BK(I+e^{AD})$ is Hurwitz, the pair (A,B) has to be controllable and the matrix $I+e^{AD}$ invertible. Since the matrices A and $I+e^{AD}$ commute, this is equivalent with the controllability condition of the pair $(A,(I+e^{-AD})B)$, which is the controllability condition in [1]. An example where the controllability condition from [1] fails, is when the matrix $I+e^{AD}$ is identically zero. This is the case for example of a second order oscillator with frequency π and input delay $D=1$.

V. Conclusion

In this work, we prove exponential stability of predictor feedback in multi-input systems with distributed input delays. We reach our result by constructing a Lyapunov functional for the closed-loop system, based on novel transformations of the actuator states. Furthermore, we design an observer for multi-output systems with distributed sensor delays and prove exponential convergence of the estimation error. Finally, a detailed example is presented that illustrates the construction of the Lyapunov functional.

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Dissipativity-Based Switching Adaptive Control

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***Abstract*—This technical note introduces a new dissipativity-based switching adaptive control strategy for uncertain systems. In this approach, the parametric uncertainties in systems are not considered on continuums but studied on finite discrete sets. Differently to multiple model supervisory control, dissipativity-based switching adaptive control needs no estimation errors for each switching decision. The switching logic is designed based on the relationship between dissipativity/passivity and adaptive systems, such that in the process of switching control, a transient boundary can be guaranteed by appropriately switching the parameter estimates. This makes it possible to apply the strategy for nonlinear systems modeled in local regions in the state space. We also discuss the implementation of the new idea to general dissipative systems and feedback passive systems. An example with simulation is employed to show the effectiveness of the approach.**

***Index Terms*—Convergence, dissipativity/passivity, switching adaptive control, transient boundary, uncertain nonlinear systems.**

I. Introduction

Transient boundedness is important for both theoretical systems synthesis and practical implementation of the controllers in applications. For instance, in nonlinear systems identification and modeling, because of the inherent nonlinear property, it is usually hard to completely identify the uncertainties. So, the models of nonlinear systems we often use for control design are locally valid along the experienced trajectories in state space and sometimes depend on the operating situation and environments [27]. If the model is only valid in the region Ω_ζ in state space, the controlled state trajectory φ should not leave Ω_ζ. More examples

Manuscript received July 20, 2009; revised August 26, 2010, October 30, 2010; accepted December 17, 2010. Date of publication December 30, 2010; date of current version March 09, 2011. This work was supported by the Australian Research Council's Discovery funding scheme (Project FF0455875) and also by the National Natural Science Foundation of China under Grants 90816028 and 60934001. Recommended by Associate Editor J.-F. Zhang.

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Digital Object Identifier 10.1109/TAC.2010.2102490