Nikolaos Bekiaris-Liberis[1]
e-mail: nbekiari@ucsd.edu

Miroslav Krstic
e-mail: krstic@ucsd.edu

Department of Mechanical and
Aerospace Engineering,
University of California,
San Diego, CA 92093-0411

Compensation of Time-Varying Input and State Delays for Nonlinear Systems

We consider general nonlinear systems with time-varying input and state delays for which we design predictor-based feedback controllers. Based on a time-varying infinite-dimensional backstepping transformation that we introduce, our controller achieves global asymptotic stability in the presence of a time-varying input delay, which is proved with the aid of a strict Lyapunov function that we construct. Then, we "backstep" one time-varying integrator and we design a globally stabilizing controller for nonlinear strict-feedback systems with time-varying delays on the virtual inputs. The main challenge in this case is the construction of the backstepping transformations since the predictors for different states use different prediction windows. Our designs are illustrated by three numerical examples, including unicycle stabilization. [DOI: 10.1115/1.4005278]

1 Introduction

Ignoring the effect of a time-varying state or input delay can have catastrophic consequences to a system [1]. Moreover, time-varying delays are present in numerous real-world applications. Successful application of various control design methods to networked control systems is constrained by the presence of time-varying input delays [2–5]. Time-varying delays appear also in supply networks, [6] and [7] and irrigation channels [8]. Last but not least, the reaction time of a driver varies over time [9,10], and hence, can be modeled as a time-varying input delay.

Several existing techniques for compensating constant input delays in linear plants, such as [11–17], are extensions of the "Smith Predictor" [18]. Extensions of these designs to linear systems with simultaneous input and state delay also exist [19–22]. In addition, predictor-based adaptive schemes for both unknown plant parameters [23,24] and delays [25–27] are available. Control techniques for nonlinear systems with input [28–30] or state delays [31–35] are recently developed. Although numerous results deal with plants with constant input delays, the problem of *compensation* of time-varying input delays, even for linear systems, is tackled in only a few references [11,36–38]. Even more rare are papers that deal with the compensation of a time-varying input delay in nonlinear systems [33]. In this paper, we continue the efforts on the compensation of input delays for nonlinear systems, initiated at Ref. [28] for constant delays, to the case of time-varying input delays. We also extend the results from Ref. [19] in two major directions: (1) In contrast with the result from Ref. [19], we consider here nonlinear plants and (2) we are dealing with time-varying instead of constant delays. These extensions are not trivial, since for nonlinear systems with time-varying input and state delays the definition of the predictor states, as well as the form of the control law, does not follow in an obvious way from the delay-free plant.

We introduce a methodology for compensating time-varying state or input delays for nonlinear systems. The assumptions that we make for the case of nonlinear plants with time-varying input delay are rather plausible: We assume that the systems under consideration are forward complete, that is to say, they cannot escape to infinity in finite time. We also assume that our plant can be asymptotically stabilized by a (possibly time-varying) control law in the absence of the delay. Based on these assumptions and by using a backstepping transformation we construct a predictor-based compensator. Our design achieves asymptotic stability which is proved using a Lyapunov functional that we construct. Then, we "backstep" one time-varying integrator and we consider nonlinear systems in the strict-feedback form with a chain of time-varying integrators. We employ an infinite-dimensional backstepping procedure and we derive a control law that uses the predicted values of the states on different predicted intervals for each state. We consider the second order case but the result can be extended recursively to the general nth-order strict-feedback class with delays in the integrator chain as well as delays on other states in the system. Finally, we illustrate our design with three numerical examples. A second order strict-feedforward system with time-varying input delay, unicycle stabilization subject to a time-varying input delay and a second order strict-feedback system with time-varying state delay.

In Sec. 2 we introduce a predictor-based delay compensation design for general stabilizable nonlinear systems. In Sec. 3 we analyze the stability properties of the closed-loop system. We extend our design methodology to the case of strict-feedback systems with time-varying delayed integrators in Sec. 4. Finally, in Sec. 5 we demonstrate our design with two numerical examples.

Notation: For a function $r(x,t):[0,1]\times[0,\infty)\mapsto\mathbb{R}$, we denote its derivative with respect to x with $r_x(x,t)$ and its derivative with respect to t as $r_t(x,t)$.

2 Problem Formulation and Controller Design

We consider the following nonlinear system with time-varying input delay

$$\dot{X}(t)=f(X(t),U(t-D(t))) \quad (1)$$

where $X\in\mathbb{R}^n$, $U\in\mathbb{R}$, $t\in\mathbb{R}_+$ and f: $f: C^1(\mathbb{R}^n\times\mathbb{R};\mathbb{R}^n)$ with $f(0,0)=0$. Throughout the paper we make the following assumptions concerning the plant (1):

Assumption 1. *The plant $\dot{X}=f(X,\omega)$ is strongly forward complete, that is, there exists a smooth positive definite function R and class $\mathcal{K}_\infty$ functions δ_1, δ_2, and δ_3 such that the following holds*

$$\delta_1(|X|)\le R(X)\le\delta_2(|X|) \quad (2)$$

$$\frac{\partial R(X)}{\partial X}f(X,\omega)\le R(X)+\delta_3(|\omega|) \quad (3)$$

[1]Corresponding author.

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for all $X \in \mathbb{R}^n$ *and for all* $\omega \in \mathbb{R}$.

Assumption 1 guarantees that system (1) does not exhibit finite escape time, i.e., for every initial condition and every bounded input signal the corresponding solution is defined for all $t \geq 0$, i.e., the maximal interval of existence is $T_{\max} = \infty$. The definition of forward-completeness is the one from [39].

Assumption 2. *There exists a feedback law* $\kappa \in C^1(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R})$ *such that the plant* $\dot{X}(t) = f(X(t), \kappa(t, X(t)) + \omega(t))$ *satisfies the uniform in time input-to-state stability property with respect to* ω *and the function* κ *is uniformly bounded with respect to its first argument, that is, there exists a class* $\mathcal{K}_\infty$ *function* $\hat{\rho}$ *such that*

$$|\kappa(t, \xi)| \leq \hat{\rho}(|\xi|), \quad \textit{for all} \quad t \geq 0 \tag{4}$$

Naturally, the time-varying delay should be uniformly bounded and positive, and hence, there must exist a positive constant m such that $m > D(t) > 0$ for all $t \geq 0$. As it turns out later on, our predictor-based compensator makes use of the inverse of the function

$$\phi(t) = t - D(t) \tag{5}$$

namely $\phi^{-1}(t)$. Hence, we have to impose certain conditions on $\phi'(t)$ that guarantee the invertibility of $\phi(t)$. We make the following assumptions regarding the function $\phi(t)$:

Assumption 3. *The function* $\phi(t) = t - D(t)$ *satisfies*

$$\phi(t) < t, \quad \textit{for all} \quad t \geq 0 \tag{6}$$

and

$$\pi_0^* = \frac{1}{\sup_{\theta \geq \phi^{-1}(0)} (\theta - \phi(\theta))} > 0 \tag{7}$$

Assumption 4. *The derivative of the function* $\phi(t)$ *satisfies*

$$\phi'(t) > 0, \quad \textit{for all} \quad t \geq 0 \tag{8}$$

and it holds that

$$\pi_1^* = \frac{1}{\sup_{\theta \geq \phi^{-1}(0)} \phi'(\theta)} > 0. \tag{9}$$

The delay time $D(t)$ has to be positive for all $t \geq 0$ (in order to guarantee casualty of the plant) which is guaranteed with condition (6) (note that $t - \phi(t) = D(t)$) in assumption 3. Moreover, relation (7) guarantees that $D(t)$ is uniformly bounded from above which implies that the control signal eventually reaches the plant. Relation (8) (or equivalently $(dD(t))/(dt) < 1$) in assumption 4 guarantees that there exists a time at which the controller kicks-in (i.e., there exists a solution to the equation $\phi(t) = 0$) and the bound (9) guarantees that $(dD(t))/(dt)$ is bounded from below.

The controller for the plant (1) that compensates the time-varying input delay and achieves global asymptotic stability of the closed-loop system is given by

$$U(t) = \kappa\left(\phi^{-1}(t), P(t)\right) \tag{10}$$

where $P(t)$ is given from

$$\begin{aligned} P(\theta) &= \left(\phi^{-1}(t) - t\right) \int_0^{\frac{\phi^{-1}(\theta)-t}{\phi^{-1}(t)-t}} f\left(P\left(\phi\left(t + y\left(\phi^{-1}(t) - t\right)\right)\right)\right. \\ &\quad \left. \times U\left(\phi\left(t + y\left(\phi^{-1}(t) - t\right)\right)\right)\right) dy + X(t) \\ &= \int_{\phi(t)}^{\theta} f(P(\sigma), U(\sigma)) \frac{d\sigma}{\phi'\left(\phi^{-1}(\sigma)\right)} + X(t), \quad \phi(t) \leq \theta \leq t \end{aligned} \tag{11}$$

for $\theta = t$. The initial condition of Eq. (11) is given for $t = 0$ as

$$P(\theta) = \int_{\phi(0)}^{\theta} f(P(\sigma), U(\sigma)) \frac{d\sigma}{\phi'\left(\phi^{-1}(\sigma)\right)} + X(0) \quad \text{for all} \quad \theta \in [\phi(0), 0] \tag{12}$$

For an implementation of the controller (10) one has to numerically integrate the finite intervals in Eqs. (11) and (12) using one of the numerical quadratures. In our simulations, we use the composite left-endpoint rectangle rule. Note that the interval of integration, i.e., the interval $t - \phi(t) = D(t)$, at each time step may not be an integer multiple of the discretization step h. In this case, the total number of points that are used in the computation of the integral are derived by rounding the number $(t - \phi(t))/h$ to the nearest integer from below. However, the study of the complexity of the algorithm in an actual implementation is beyond the scope of this paper which concentrates in basic continuous-time designs.

The quantity P(t) given in (predictor) is the $\phi^{-1}(t) - t$ time units ahead predictor of $X(t)$, which can be seen as follows: Differentiating relation (predictor) with respect to θ, setting $\theta = t$ and performing a change of variables $\tau = \phi^{-1}(t)$ in the ODE for $X(\tau)$ given in Eq. (1) (where t is replaced by τ), one observes that $P(t)$ satisfies the same ODE in t as $X(\phi^{-1}(t))$. Since from Eq. (12) it follows that $P(\phi(0)) = X(0)$, we have that $P(0) = X(\phi^{-1}(0))$. Hence, indeed $P(t) = X(\phi^{-1}(t))$ for all $t \geq 0$.

The choice of (predictor) becomes clear by considering the case where the input delay is constant, of length, say, D. In this case the predictor that corresponds to (predictor) is

$$\begin{aligned} P_c(t) &= D \int_0^1 f(P_c(t + D(y-1)), U(t + D(y-1))) dy + X(t) \\ &= \int_{t-D}^{t} f(P_c(\theta), U(\theta)) d\theta + X(t) \end{aligned} \tag{13}$$

The signal $P_c(t)$ in Eq. (13) is the D time units ahead predictor of $X(t)$ (see also Ref. [28]), i.e., $P_c(t) = X(t + D)$ for all $t \geq 0$. The relationship between (predictor) and Eq. (13) can be explained as follows: Let us define the predictor time in the case of a time-varying delay as $\phi^{-1}(t) - t$, which is uniformly bounded based on relation (9). Analogously, for the case of a constant delay define the predictor time as $\phi_c^{-1}(t) - t$ which of course equals D. One can now re-write Eq. (13) by substituting D with $\phi_c^{-1}(t) - t$ as

$$\begin{aligned} P_c(t) &= D \int_0^1 f(P_c(t + D(y-1)), U(t + D(y-1))) dy + X(t) \\ &= \left(\phi_c^{-1}(t) - t\right) \int_0^1 f(P_c(t + D(y-1)), U(t + D(y-1))) dy + X(t) \end{aligned} \tag{14}$$

This is the exact analog of relation (predictor) for the constant delay case. Moreover, the choice of the function $\phi(t + y(\phi^{-1}(t) - t))$ inside the integral in equation (predictor) is guided by the choice of an appropriate state, namely $U(\phi(t + y(\phi^{-1}(t) - t)))$ for all $y \in [0, 1]$, for the infinite-dimensional state of the actuator that also makes $U(\phi(t + y(\phi^{-1}(t) - t)))$ equal to $U(t)$ in the case where $y = 1$ and equal to $U(\phi(t))$ in the case where $y = 0$.

3 Stability Analysis

In this section we prove the following result:

Theorem 1. *Consider the system* (1) *together with the control law* (10)–(12). *Under assumptions 1–4 there exists a class* $\mathcal{KL}$ *function* α *such that the following holds*

$$|X(t)| + \sup_{\phi(t)\le\theta\le t} |U(\theta)| \le \alpha\left(|X(0)| + \sup_{\phi(0)\le\theta\le 0} |U(\theta)|, t\right), \quad t \ge 0 \tag{15}$$

We prove the above theorem using a series of technical lemmas. We first introduce an equivalent representation of the plant (1) using a transport partial differential equation (PDE) representation for the actuator state as

$$\dot{X}(t) = f(X(t), \quad u(0,t)) \tag{16}$$

$$u_t(x,t) = \pi(x,t)u_x(x,t), \quad x \in [0,1] \tag{17}$$

$$u(1,t) = U(t) \tag{18}$$

where

$$\pi(x,t) = \frac{1 + x\left(\frac{d(\phi^{-1}(t))}{dt} - 1\right)}{\phi^{-1}(t) - t} \tag{19}$$

and $\phi(t)$ is defined in Eq. (5). The choice of the transport speed $\pi(x,t)$ is guided by the fact that we seek a representation for the infinite-dimensional actuator state $u(x,t)$ such that relations (18) and

$$u(0,t) = U(\phi(t)) \tag{20}$$

are satisfied. One can verify that $u(x,t)$ is given by

$$u(x,t) = U\left(\phi\left(t + x\left(\phi^{-1}(t) - t\right)\right)\right) \tag{21}$$

and consequently both Eqs. (18) and (20) are satisfied. We give first an alternative proof of the fact that $P(t)$ is the $\phi^{-1}(t) - t$ time units ahead predictor of $X(t)$, based on the PDE representation (16)–(18).

Lemma 1. *The signal* P(t) *in Eq.* (11) *is the* $\phi^{-1}(t) - t$ *time units ahead predictor of* $X(t)$, *i.e., it holds that*

$$P(t) = X(\phi^{-1}(t)), \quad \textit{for all} \quad t \ge 0 \tag{22}$$

Furthermore, an equivalent representation for $P(t)$ *is as*

$$p(1,t) = \left(\phi^{-1}(t) - t\right)\int_0^1 f(p(y,t), u(y,t))dy + X(t) \tag{23}$$

where

$$p(x,t) = P\left(\phi\left(t + x\left(\phi^{-1}(t) - t\right)\right)\right) \tag{24}$$

Proof. Consider

$$p(x,t) = \left(\phi^{-1}(t) - t\right)\int_0^x f(p(y,t), u(y,t))dy + X(t), \quad x \in [0,1] \tag{25}$$

Differentiating the above relation with respect to time and using Eq. (17) we get that

$$\begin{aligned} p_t(x,t) &= \left(\phi^{-1}(t) - t\right)\int_0^x \frac{\partial f(p(y,t), u(y,t))}{\partial p(y,t)} p_t(y,t) \\ &\quad + \left(\phi^{-1}(t) - t\right)\int_0^x \frac{\partial f(p(y,t), u(y,t))}{\partial u(y,t)} \pi(y,t)u_y(y,t)dy \\ &\quad + \left(\frac{d\phi^{-1}(t)}{dt} - 1\right)\int_0^x f(p(y,t), u(y,t))dy + f(p(0,t), u(0,t)) \end{aligned} \tag{26}$$

Moreover, differentiating (25) with respect to the spatial variable x we have

$$\begin{aligned} \pi(x,t)p_x(x,t) &= \left(\phi^{-1}(t) - t\right)\pi(x,t)f(p(x,t), u(x,t)) \\ &= \left(\phi^{-1}(t) - t\right)\int_0^x \frac{d(\pi(y,t)f(p(y,t), u(y,t)))}{dy}dy \\ &\quad + \left(\phi^{-1}(t) - t\right)\pi(0,t)f(p(0,t), u(0,t)) \\ &= \left(\phi^{-1}(t) - t\right)\int_0^x \pi_y(y,t)f(p(y,t), u(y,t))dy \\ &\quad + \left(\phi^{-1}(t) - t\right)\int_0^x \pi(y,t)\frac{\partial f(p(y,t), u(y,t))}{\partial p(y,t)} p_y(y,t)dy \\ &\quad + \left(\phi^{-1}(t) - t\right)\int_0^x \pi(y,t)\frac{\partial f(p(y,t), u(y,t))}{\partial u(y,t)} u_y(y,t)dy \\ &\quad + \left(\phi^{-1}(t) - t\right)\pi(0,t)f(p(0,t), u(0,t)) \end{aligned} \tag{27}$$

Comparing Eq. (26) with Eq. (27) and using the facts that

$$\pi(0,t) = \frac{1}{\phi^{-1}(t) - t} \quad \text{and} \quad \pi_x(x,t) = \frac{\frac{d\phi^{-1}(t)}{dt} - 1}{\phi^{-1}(t) - t}$$

which follow from the definition of $\pi(x,t)$ in Eq. (19) we get that

$$\begin{aligned} p_t(x,t) - \pi(x,t)p_x(x,t) &= \left(\phi^{-1}(t) - t\right)\int_0^x \frac{\partial f(p(y,t), u(y,t))}{\partial p(y,t)}(p_t(y,t) \\ &\quad - \pi(y,t)p_y(y,t))dy \end{aligned} \tag{28}$$

Define now the function $G(x,t) = p_t(x,t) - \pi(x,t)p_x(x,t)$, which satisfies

$$G_x(x,t) = \left(\phi^{-1}(t) - t\right)\frac{\partial f(p(x,t), u(x,t))}{\partial p(x,t)} G(x,t) \tag{29}$$

$$G(0,t) = 0 \tag{30}$$

Hence, $G(x,t) = 0$, for all $x \in [0,1]$. Equivalently,

$$p_t(x,t) = \pi(x,t)p_x(x,t), \quad \text{for all} \quad x \in [0,1] \tag{31}$$

Using the above relation and by defining $p(1,t) = P(t)$, we get Eqs. (23) and (24). Moreover, since from Eq. (25) it holds that $p(0,t) = X(t)$, using relation (24) we get Eq. (22).

We next transform our original system given in Eqs. (16)–(18) to the "target system" which we prove later on to be globally asymptotically stable.

Lemma 2. *The infinite-dimensional transformation of the actuator state defined by*

$$w(x,t) = u(x,t) - \kappa(r(x,t), p(x,t)) \tag{32}$$

where

$$r(x,t) = t + x\left(\phi^{-1}(t) - t\right) \tag{33}$$

together with the control law given in Eq. (10) *transforms the system* (16)–(18) *to the "target system" given by*

$$\dot{X}(t) = f(X(t), \kappa(t, X(t)) + w(0,t)) \tag{34}$$

$$w_t(x,t) = \pi(x,t)w_x(x,t), \quad x \in [0,1] \tag{35}$$

$$w(1,t) = 0 \tag{36}$$

Proof. We first point out that $r(x,t)$ in Eq. (33) satisfies

$$r_t(x,t) = \pi(x,t)r_x(x,t) \tag{37}$$

$$r(0,t) = t \tag{38}$$

From Eq. (32) and by using the chain rule together with relations (17), (31), and (37) we get Eq. (35). Using Eqs. (32) and (33) for $x=0$ and $x=1$ together with Eq. (10) we arrive at Eqs. (34) and (36).

We now define the inverse of the transformation (32).

Lemma 3. *The inverse of the infinite-dimensional transformation defined in Eq. (32) is given by*

$$u(x,t) = w(x,t) + \kappa(r(x,t), \xi(x,t)) \tag{39}$$

where $\xi(x,t)$ *is defined as*

$$\xi(x,t) = \left(\phi^{-1}(t) - t\right)\int_0^x f(\xi(y,t), \kappa(r(y,t), \xi(y,t)) + w(y,t))dy + X(t) \tag{40}$$

Proof. We first point out that $\xi(x,t)$ is for the closed-loop system (34)–(36), what $p(x,t)$ is for system (16)–(18). Although

$$\xi(x,t) = p(x,t), \quad \text{for all} \quad x \in [0,1] \tag{41}$$

$\xi(x,t)$ is driven by the transformed input $w(x,t)$, whereas $p(x,t)$ is driven by the input $u(x,t)$. In other words, the direct transformation is defined as $(X(t), u(x,t)) \mapsto (X(t), w(x,t))$ and is given in Eq. (32), where $p(x,t)$ is given as a function of $X(t)$ and $u(x,t)$ through relation (25). Analogously, the inverse transformation is defined as $(X(t), w(x,t)) \mapsto (X(t), u(x,t))$ and is given in Eq. (39) where $\xi(x,t)$ is given as a function of $X(t)$ and $w(x,t)$ through relation (40).

We prove next stability of the "target system" (34)–(36).

Lemma 4. *There exist a* $\mathcal{KL}$ *class function* β *such that*

$$|X(t)| + \sup_{\phi(t)\le\theta\le t}|W(\theta)| \le \beta\left(|X(0)| + \sup_{\phi(0)\le\theta\le 0}|W(\theta)|, t\right), \quad t \ge 0 \tag{42}$$

Proof. Based on assumption 2 and [40] we can conclude that there exist a smooth positive definite function $S(X(t))$ and class $\mathcal{K}$ functions γ_1, γ_2, γ_3, and γ_4 such that

$$\gamma_1(|X(t)|) \le S(X(t)) \le \gamma_2(|X(t)|) \tag{43}$$

$$\frac{\partial S(X(t))}{\partial X(t)} f(X(t), \kappa(t, X(t)) + w(0,t)) \le -\gamma_3(|X(t)|) + \gamma_4(|w(0,t)|) \tag{44}$$

Consider now the functional

$$\begin{aligned} L(t) &= \sup_{x\in[0,1]}|e^{cx}w(x,t)| \\ &= \lim_{n\to\infty}\left(\int_0^1 e^{2ncx}w^{2n}(x,t)dx\right)^{1/(2n)} \end{aligned} \tag{45}$$

Following the calculations in Ref. [28] and in Ref. [36] one concludes that

$$\dot{L}(t) \le -c\pi_0^* \min\{1, \pi_1^*\}L(t) \tag{46}$$

Consider now the following Lyapunov function for system (34)–(36) which is positive definite and radially unbounded

$$V(t) = S(X(t)) + \frac{2}{c\pi_0^*\min\{1,\pi_1^*\}}\int_0^{L(t)}\frac{\gamma_4(r)}{r}dr \tag{47}$$

Taking the time derivative of $V(t)$ along the solutions of the "target system" Eqs. (34)–(36) and using Eqs. (44) and (46) we have $\dot{V}(t) \le -\gamma_3(|X(t)|) + \gamma_4(|w(0,t)|) - 2\gamma_4(L(t))$. Using the fact that $\gamma_4(|w(0,t)|) \le \gamma_4\left(\sup_{x\in[0,1]}|e^{cx}w(x,t)|\right) = \gamma_4(L(t))$, we get $\dot{V}(t) \le -\gamma_3(|X(t)|) - \gamma_4(L(t))$. Using Eq. (43), the definition of $L(t)$ in Eqs. (45) and (47) we conclude that there exists a class $\mathcal{K}$ function γ_5 such that

$$\dot{V}(t) \le -\gamma_5(V(t)) \tag{48}$$

Consequently, using the comparison principle and Lemma 4.4 in [41] we can conclude that there exist a class $\mathcal{KL}$ function β_1 such that

$$V(t) \le \beta_1(V(0), t) \tag{49}$$

Using Eq. (43), the definition of $V(t)$ in Eq. (47) and the properties of a class $\mathcal{K}$ function we arrive at

$$|X(t)| + L(t) \le \beta_2(|X(t)| + L(0), t) \tag{50}$$

for some class $\mathcal{KL}$ function β_2. Moreover, from Eq. (45) we get

$$\sup_{x\in[0,1]}|w(x,t)| \le L(t) \le e^c \sup_{x\in[0,1]}|w(x,t)| \tag{51}$$

Define now the solution of the transport PDE (35)–(36) as

$$w(x,t) = W\left(\phi\left(t + x\left(\phi^{-1}(t) - t\right)\right)\right) \tag{52}$$

Then combining Eqs. (51) and (52) we get

$$\sup_{\theta\in[\phi(t),t]}|W(\theta)| \le L(t) \tag{53}$$

$$L(0) \le e^c \sup_{\theta\in[\phi(0),0]}|W(\theta)| \tag{54}$$

Combining Eqs. (50), (53), and (54) the lemma is proved for some class $\mathcal{KL}$ function β.

The following two lemmas allows one to establish stability in the original variables.

Lemma 5. *There exist a class* $\mathcal{K}_\infty$ *function* ρ_1 *such that*

$$\sup_{x\in[0,1]}|p(x,t)| \le \rho_1\left(|X(t)| + \sup_{x\in[0,1]}|u(x,t)|\right), \quad t \ge 0 \tag{55}$$

Proof. Consider the system

$$\dot{Y}(\tau) = f(Y(\tau), \omega(\tau)) \tag{56}$$

with μ being a positive constant. With a time rescaling, namely $t = \frac{\tau}{\mu}$, we get

$$\frac{dY(t\mu)}{dt} = \mu f(Y(t\mu), \omega(t\mu)) \tag{57}$$

Under assumption 1 we get

$$\frac{\partial R(Y'(t))}{\partial Y'(t)}\mu f(Y'(t), \omega'(t)) \le \mu(R(Y'(t)) + \delta_3(|\omega'(t)|)) \tag{58}$$

where

$$Y'(t) = Y(t\mu) \tag{59}$$

$$\omega'(t) = \omega(t\mu) \tag{60}$$

By differentiating relation (25) with respect to the spatial variable x we get the following ordinary differential equation (ODE) in x

$$p_x(x,t) = \left(\phi^{-1}(t) - t\right) f\left(p(x,t), u(x,t)\right) \tag{61}$$

with initial conditions

$$p(0,t) = X(t) \tag{62}$$

Viewing $\phi^{-1}(t) - t$ in Eq. (61) as a parameter (rather as the running variable) and comparing Eq. (61) with Eq. (57), with the help of Eq. (58) we get

$$\begin{aligned} &\frac{\partial R(p(x,t))}{\partial p(x,t)}\left(\phi^{-1}(t) - t\right) f\left(p(x,t), u(x,t)\right) \\ &\le \left(\phi^{-1}(t) - t\right)\left(R(p(x,t)) + \delta_3(|u(x,t)|)\right) \end{aligned} \tag{63}$$

Using the comparison principle and relation (62) we have that

$$\begin{aligned} R(p(x,t)) &\le e^{x\left(\phi^{-1}(t)-t\right)} R(X(t)) + \left(\phi^{-1}(t) - t\right) \\ &\int_0^x e^{\left(\phi^{-1}(t)-t\right)(x-y)} \delta_3(|u(y,t)|) dy \\ &\le e^{1/\pi_0^*}\left(R(X(t)) + \sup_{x\in[0,1]} \delta_3(|u(x,t)|) dy\right) \end{aligned} \tag{64}$$

where we used bound (7). Using Eq. (2) and the properties of class $\mathcal{K}_\infty$ functions we get the statement of the lemma.

Lemma 6. *There exist a class $\mathcal{K}_\infty$ function ρ_2 such that*

$$\sup_{x\in[0,1]} |\xi(x,t)| \le \rho_2\left(|X(t)| + \sup_{x\in[0,1]} |w(x,t)|\right), \quad t \ge 0 \tag{65}$$

Proof. Differentiating Eq. (40) with respect to x we get the following ODE in x

$$\begin{aligned} &\xi_x(x,t) = \left(\phi^{-1}(t) - t\right) f\left(\xi(x,t), \kappa(r(x,t), \xi(x,t)) + w(x,t)\right) \\ &\quad x \in [0,1] \end{aligned}$$

with initial conditions

$$\xi(0,t) = X(t) \tag{67}$$

With a rescaling of the spatial variable x, say y as

$$y = t + x\left(\phi^{-1}(t) - t\right) \tag{68}$$

and by defining

$$\zeta(y,t) = \xi\left(\frac{y-t}{\phi^{-1}(t) - t}, t\right) \tag{69}$$

$$\omega(y,t) = w\left(\frac{y-t}{\phi^{-1}(t) - t}, t\right) \tag{70}$$

with the help of Eq. (33) we rewrite Eqs. (66) and (67) in the new spatial variable y as

$$\zeta_y(y,t) = f\left(\zeta(y,t), \kappa(y, \zeta(y,t)) + \omega(y,t)\right), \quad y \in [t, \phi^{-1}(t)] \tag{71}$$

$$\zeta(t,t) = X(t) \tag{72}$$

Under assumption 2 and [40], there exist a class $\mathcal{KL}$ function $\beta_3(\cdot, y)$ and a class $\mathcal{K}$ function γ_6 such that

$$\zeta(y,t) \le \beta_3(|X(t)|, y-t) + \gamma_6\left(\sup_{y\in[t,\phi^{-1}(t)]} |\omega(y,t)|\right) \tag{73}$$

where we also used Eq. (67). Using Eqs. (68)–(70) we have that

$$\begin{aligned} &\xi(x,t) \le \beta_3\left(|X(t)|, x\left(\phi^{-1}(t) - t\right)\right) + \gamma_6\left(\sup_{x\in[0,1]} |w(x,t)|\right) \\ &\quad \forall x \in [0,1] \end{aligned} \tag{74}$$

By noting that β_3 is a decreasing function of the second argument and by taking the supremum of both sides we arrive at

$$\sup_{x\in[0,1]} \xi(x,t) \le \beta_3(|X(t)|, 0) + \gamma_6\left(\sup_{x\in[0,1]} |w(x,t)|\right) \tag{75}$$

With standard properties of class $\mathcal{K}_\infty$ and $\mathcal{KL}$ functions we get the bound of the lemma.

Proof of Theorem 1: Under assumption 2 and by using relation (32) we have

$$\sup_{x\in[0,1]} |w(x,t)| \le \sup_{x\in[0,1]} \left(|u(x,t)| + \hat{\rho}(|p(x,t)|)\right) \tag{76}$$

$$\sup_{x\in[0,1]} |u(x,t)| \le \sup_{x\in[0,1]} \left(|w(x,t)| + \hat{\rho}(|\xi(x,t)|)\right) \tag{77}$$

With Lemmas 5 and 6 we have that

$$\sup_{x\in[0,1]} |w(x,t)| \le \sup_{x\in[0,1]} |u(x,t)| + \hat{\rho}^\circ \rho_1\left(|X(t)| + \sup_{x\in[0,1]} |u(x,t)|\right) \tag{78}$$

$$\sup_{x\in[0,1]} |u(x,t)| \le \sup_{x\in[0,1]} |w(x,t)| + \hat{\rho}^\circ \rho_2\left(|X(t)| + \sup_{x\in[0,1]} |w(x,t)|\right) \tag{79}$$

Using Eqs. (21) and (52) one can conclude that there exist class $\mathcal{K}_\infty$ functions $\hat{\alpha}$ and $\hat{\beta}$ such that

$$|X(t)| + \sup_{\phi(t)\le\theta\le t} |W(\theta)| \le \hat{\alpha}\left(|X(t)| + \sup_{\phi(t)\le\theta\le t} |U(\theta)|\right) \tag{80}$$

$$|X(t)| + \sup_{\phi(t)\le\theta\le t} |U(\theta)| \le \hat{\beta}\left(|X(t)| + \sup_{\phi(t)\le\theta\le t} |W(\theta)|\right) \tag{81}$$

From Lemma 4 and Eq. (81) we get

$$|X(t)| + \sup_{\phi(t)\le\theta\le t} |U(\theta)| \le \hat{\beta}^\circ \beta\left(|X(0)| + \sup_{\phi(0)\le\theta\le 0} |W(\theta)|, t\right) \tag{82}$$

Setting $t = 0$ in Eq. (80), Theorem 1 is proved with $\alpha = \hat{\beta}^\circ \beta^\circ \hat{\alpha}$.

4 Adding a Time-Varying Delayed Integrator

In this section, we extend the design of the previous sections to nonlinear strict-feedback systems with time-varying delayed integrators. We consider the following system

$$\dot{X}_1(t) = f_1(X_1(t)) + X_2(\phi_1(t)) \tag{83}$$

$$\dot{X}_2(t) = f_2(X_1(t), X_2(t)) + U(\phi_2(t)) \tag{84}$$

where we assume for notational simplicity that X_1 and $X_2 \in \mathbb{R}$, $U \in \mathbb{R}$, and $t \in \mathbb{R}_+$. Moreover, it is assumed that the functions $\phi_1(t)$ and $\phi_2(t)$ satisfy assumptions 3 and 4 with

$$\pi^*_{1_{\phi_1}} = \frac{1}{\sup_{\theta\geq\phi_1^{-1}(0)} \phi_1'(\theta)} > 0 \quad (85)$$

$$\pi^*_{1_{\phi_2}} = \frac{1}{\sup_{\theta\geq\phi_2^{-1}(0)} \phi_2'(\theta)} > 0 \quad (86)$$

$$\pi^*_{0_{\phi_1}} = \frac{1}{\sup_{\theta\geq\phi_1^{-1}(0)} (\theta-\phi_1(\theta))} > 0 \quad (87)$$

$$\pi^*_{0_{\phi_2}} = \frac{1}{\sup_{\theta\geq\phi_2^{-1}(0)} (\theta-\phi_2(\theta))} > 0 \quad (88)$$

$$\pi^*_{1_{\psi}} = \frac{1}{\sup_{\theta\geq\psi^{-1}(0)} \psi'(\theta)} > 0 \quad (89)$$

$$\pi^*_{0_{\psi}} = \frac{1}{\sup_{\theta\geq\psi^{-1}(0)} (\theta-\psi(\theta))} > 0 \quad (90)$$

where

$$\psi(t) = \phi_2(\phi_1(t)) \quad (91)$$

We assume that $f_1 : C^1(\mathbb{R};\mathbb{R})$, $f_2 : \mathbb{R}^2 \mapsto \mathbb{R}$ is locally Lipschitz with respect to its arguments and it holds that $f_1(0)=0$, $f_2(0,0)=0$. For system (83) and (84) we design a predictor-based feedback for stabilizing the origin. System (83) and (84) can possibly escape to infinity before the controller reaches it. We thus make the following assumption.

Assumption 5. *System* (83) and (84) *is forward-complete*.

The predictor-based controller for system (83) and (84) is

$$U(t) = -f_2\left(P_1\left(\psi\left(\phi_2^{-1}(t)\right)\right), P_2(t)\right) - c_2(P_2(t)+c_1P_1(t)+f_1(P_1(t))) - \left(c_1+\frac{\partial f_1(P_1)}{\partial P_1}\right)(f_1(P_1(t))+P_2(t))R\left(\phi_2^{-1}(t)\right) \quad (92)$$

where c_1, c_2 are positive constants that satisfy $c_1\pi^*_{1_{\phi_1}} \neq 2c_2$ (a choice made to simplify one step in the analysis) and

$$\begin{aligned} P_1(t) &= X_1(t) + \left(\psi^{-1}(t)-t\right)\int_0^1 \left(f_1\left(P_1\left(\psi\left(t+y\left(\psi^{-1}(t)-t\right)\right)\right)\right)\right. \\ &\quad \left.+ P_2\left(\psi\left(t+\left(\psi^{-1}(t)-t\right)y\right)\right)\right)dy \\ &= X_1(t) + \int_{\psi(t)}^t (f_1(P_1(\theta))+P_2(\theta))\frac{d\theta}{\psi'\left(\psi^{-1}(\theta)\right)} \end{aligned} \quad (93)$$

$$\begin{aligned} P_2(t) &= X_2(t) + \left(\phi_2^{-1}(t)-t\right)\int_0^1 \left(f_2\left(P_1\left(\psi\left(t+y\left(\phi_2^{-1}(t)-t\right)\right)\right)\right)\right. \\ &\quad P_2\left(\phi_2\left(t+y\left(\phi_2^{-1}(t)-t\right)\right)\right)\Big) + U\left(\phi_2\left(t+y\left(\phi_2^{-1}(t)-t\right)\right)\right)\Big)dy \\ &= X_2(t) + \int_{\phi_2(t)}^t \frac{\left(f_2\left(P_1\left(\psi\left(\phi_2^{-1}(\theta)\right)\right), P_2(\theta)\right)+U(\theta)\right)d\theta}{\phi_2'\left(\phi_2^{-1}(\theta)\right)} \end{aligned} \quad (94)$$

$$R(t) = \frac{d\phi_1^{-1}(t)}{dt} \quad (95)$$

with initial conditions as

$$P_1(\theta) = X_1(0) + \int_{\psi(0)}^{\theta} (f_1(P_1(\sigma))+P_2(\sigma))\frac{d\sigma}{\psi'\left(\psi^{-1}(\sigma)\right)}, \quad \theta\in[\psi(0),0] \quad (96)$$

$$P_2(\theta) = X_2(0) + \int_{\phi_2(0)}^{\theta} \frac{\left(f_2\left(P_1\left(\psi\left(\phi_2^{-1}(\sigma)\right)\right), P_2(\sigma)\right)+U(\sigma)\right)d\sigma}{\phi_2'\left(\phi_2^{-1}(\sigma)\right)}, \quad \theta\in[\phi_2(0),0] \quad (97)$$

We now state the following theorem that is concerned with the stability properties of the closed-loop system that is comprised of the plant (83) and (84) with the controller given in Eq. (92).

Theorem 2. *Consider the plant* (83) and (84) *together with the controller* (92)–(97). *Under assumptions 3–5 there exists a class* $\mathcal{KL}$ *function* β_4 *such that for all* $X_1(0)\in\mathbb{R}$ *and for all continuous initial conditions* $X_2(s), \phi_1(0)\leq s\leq 0$ *and* $U(s), \phi_2(0)\leq s\leq 0$ *the following holds*

$$|X_1(t)| + \|X_2(t)\|_\infty + \|U(t)\|_\infty \leq \beta_4(|X_1(0)| + \|X_2(0)\|_\infty + \|U(0)\|_\infty, t), \quad t\geq 0 \quad (98)$$

where

$$\|U(t)\|_\infty = \sup_{\theta\in[\phi_2(t)-t,0]} |U(t+\theta)| \quad (99)$$

$$\|X_2(t)\|_\infty = \sup_{\theta\in[\phi_1(t)-t,0]} |X_2(t+\theta)| \quad (100)$$

We prove this theorem using a series of technical lemmas.

Lemma 7. *The signals* $P_1(t)$ *and* $P_2(t)$ *defined in Eq.* (93) *and in Eq.* (94) *are the* $\psi^{-1}(t)-t$ *and* $\phi_2^{-1}(t)-t$ *time units ahead predictors of the* $X_1(t)$ *and* $X_2(t)$, *respectively. Moreover, an equivalent representation for Eqs.* (93) *and* (94) *is given by*

$$\begin{aligned} p_1(1,t) &= X_1(t) + \left(\psi^{-1}(t)-t\right)\int_0^1 \left(f_1(p_1(y,t))\right. \\ &\quad \left.+ p_2\left(\frac{\phi_1\left(t+\left(\psi^{-1}(t)-t\right)y\right)-t}{\phi_2^{-1}(t)-t}, t\right)\right)dy \end{aligned} \quad (101)$$

$$p_2(1,t) = X_2(t) + \left(\phi_2^{-1}(t)-t\right)\int_0^1 \left(f_2\left(p_1\left(\frac{\phi_2^{-1}(t)-t}{\psi^{-1}-t}y, t\right), p_2(y,t)\right) + u(y,t)\right)dy \quad (102)$$

where

$$p_1(x,t) = P_1\left(\psi\left(t+x\left(\psi^{-1}(t)-t\right)\right)\right) \quad (103)$$

$$p_2(x,t) = P_2\left(\phi_2\left(t+x\left(\phi_2^{-1}(t)-t\right)\right)\right) \quad (104)$$

and x varies in $[0,1]$.

Proof. Consider the equivalent representation of system (83) and (84) using transport PDEs for the delayed state $X_2(t)$ and the controller $U(t)$ as

$$\dot{X}_1(t) = f_1(X_1(t)) + \xi_2(0,t) \quad (105)$$

$$\xi_{2_t}(x,t) = \pi_1(x,t)\xi_{2_x}(x,t), \quad x\in[0,1] \quad (106)$$

$$\xi_2(1,t) = X_2(t) \quad (107)$$

$$\dot{X}_2(t) = f_2(X_1(t), X_2(t)) + u(0,t) \quad (108)$$

$$u_t(x,t) = \pi_2(x,t)u_x(x,t), \quad x\in[0,1] \quad (109)$$

$$u(1,t) = U(t) \quad (110)$$

with

$$\pi_1(x,t) = \frac{1 + x\left(\frac{d\left(\phi_1^{-1}(t)\right)}{dt} - 1\right)}{\phi_1^{-1}(t)-t} \quad (111)$$

$$\pi_2(x,t) = \frac{1 + x\left(\frac{d(\phi_2^{-1}(t))}{dt} - 1\right)}{\phi_2^{-1}(t) - t} \tag{112}$$

Consider the following ODEs in x (to become clear that these are ODEs in x, one should view the time t as a parameter rather than as the running variable of the ODE),

$$p_{1_x}(x,t) = (\psi^{-1}(t) - t) \times \left(f_1(p_1(x,t)) + p_2\left(\frac{\phi_1(t + (\psi^{-1}(t) - t)x) - t}{\phi_2^{-1}(t) - t}, t\right)\right) \tag{113}$$

$$p_{2_x}(x,t) = (\phi_2^{-1}(t) - t) \times \left(f_2\left(p_1\left(\frac{\phi_2^{-1}(t) - t}{\psi^{-1} - t}x, t\right)\right), p_2(x,t)) + u(x,t)\right) \tag{114}$$

where, x varies in [0,1]. The initial conditions for the above system of ODEs are given by

$$p_i(0,t) = X_i(t), \quad i = 1,2 \tag{115}$$

and

$$p_2(\theta_2, t) = X_2(t + \theta_2(\phi_2^{-1}(t) - t)), \quad \theta_2 \in \left[\frac{\phi_1(t) - t}{\phi_2^{-1}(t) - t}, 0\right] \tag{116}$$

Assume for the moment that the following holds true

$$p_{1_t}(x,t) = \pi_3(x,t) p_{1_x}(x,t), \quad x \in [0,1] \tag{117}$$

$$p_{2_t}(x,t) = \pi_2(x,t) p_{2_x}(x,t), \quad x \in [0,1] \tag{118}$$

where

$$\pi_3(x,t) = \frac{1 + x\left(\frac{d(\psi^{-1}(t))}{dt} - 1\right)}{\psi^{-1}(t) - t} \tag{119}$$

Then, by taking into account Eq. (115) we have that

$$p_1(x,t) = X_1(t + x(\psi^{-1}(t) - t)), \quad x \in [0,1] \tag{120}$$

$$p_2(x,t) = X_2(t + x(\phi_2^{-1}(t) - t)), \quad x \in [0,1] \tag{121}$$

By defining

$$p_1(1,t) = P_1(t) \tag{122}$$

$$p_2(1,t) = P_2(t) \tag{123}$$

we get Eq. (103). By integrating from 0 to x (113) and (114) we get

$$p_1(x,t) = X_1(t) + (\psi^{-1}(t) - t)\int_0^x (f_1(p_1(y,t)) + p_2\left(\frac{\phi_1(t + (\psi^{-1}(t) - t)y) - t}{\phi_2^{-1}(t) - t}, t\right)\Bigg)dy \tag{124}$$

$$p_2(x,t) = X_2(t) + (\phi_2^{-1}(t) - t) \times \int_0^x \left(f_2\left(p_1\left(\frac{\phi_2^{-1}(t) - t}{\psi^{-1} - t}y, t\right)\right), p_2(y,t)) + u(y,t)\right)dy \tag{125}$$

By setting in each $p_i(x,t)$, $x = 1$ and using Eq. (103) we get Eqs. (101) and (102).

To see that Eq. (117) holds, it is sufficient to prove that Eqs. (120) and (121) are the unique solution of the ODEs in x given by Eqs. (113) and (114) with the initial conditions (115) and (116). In this case relations (117) and (118) hold. Thus, it remains to prove that Eqs. (120) and (121) are the unique solution of the initial value problem (113)–(116). Toward that end, we substitute Eqs. (120) and (121) into Eqs. (113) and (114)

$$X_1'(t + x(\psi^{-1}(t) - t)) = f_1(X_1(t + x(\psi^{-1}(t) - t))) + X_2(\phi_1(t + (\psi^{-1}(t) - t)x)) \tag{126}$$

$$X_2'(t + x(\phi_2^{-1}(t) - t)) = f_2(X_1(t + (\phi_2^{-1}(t) - t)x), \times X_2(t + x(\phi_2^{-1}(t) - t))) + U(\phi_2(t + x(\phi_2^{-1}(t) - t))) \tag{127}$$

where the prime symbol denotes the derivative with respect to the argument of a function. Taking into account Eqs. (83) and (84), we conclude that, indeed, Eqs. (120) and (121) are solution of the ODEs in x given by Eqs. (113) and (114). Furthermore, Eqs. (120) and (121) satisfy the initial conditions (115) and (116). Since $X_2(s), \phi_1(0) \le s \le 0$ and $U(s), \phi_2(0) \le s \le 0$ are continuous and based on assumption 5 and [42] (Chap. 2.2), $X_2(t + \theta_2(\phi_2^{-1}(t) - t))$ is continuous for all $\theta_2 \in [(\phi_1(t) - t)/(\phi_2^{-1}(t) - t), 0]$ and $t \ge 0$. Using Eq. [42] we conclude that Eqs. (120) and (121) is the unique solution of the ODEs in x given by Eqs. (113) and (114) with the initial conditions (115) and (116). Thus, Eqs. (117) and (118) hold.

It is important here to observe that the total delay from the input $U(t)$ to the state $X_1(t)$ is $t - \psi(t)$ and to the state $X_2(t)$ is $t - \phi_2(t)$. This explains the fact that our predictor intervals are different for each state and specifically must be $\psi^{-1}(t) - t$ for $X_1(t)$ and $\phi_2^{-1}(t) - t$ for $X_2(t)$. Our controller design is based on a recursive procedure that transforms system (105)–(110) to a target system which is globally asymptotically stable with the controller (92)–(97). Then, using the invertibility of this transformation, we prove global asymptotic stability of the original system. We now state this transformation, along with its inverse.

Lemma 8. *The state transformation defined by*

$$Z_1(t) = X_1(t) \tag{128}$$

$$Z_2(t) = X_2(t) + f_1\left(p_1\left(\frac{\mu(t)}{\lambda(t)}, t\right)\right) + c_1 p_1\left(\frac{\mu(t)}{\lambda(t)}, t\right) \tag{129}$$

where

$$\mu(t) = \phi_1^{-1}(t) - t \tag{130}$$

$$\lambda(t) = \psi^{-1}(t) - t \tag{131}$$

along with the transformation of the actuator state

$$\begin{aligned} w(x,t) = {} & u(x,t) + f_2\left(p_1\left(\frac{\rho(t)}{\lambda(t)}x, t\right), p_2(x,t)\right) \\ & + c_2\left(p_2(x,t) + f_1\left(p_1\left(\frac{\mu(x\rho(t) + t) + x\rho(t)}{\lambda(t)}, t\right)\right)\right. \\ & \left. + c_1 p_1\left(\frac{\mu(x\rho(t) + t) + x\rho(t)}{\lambda(t)}, t\right)\right) \\ & + \left(\frac{\partial f_1\left(p_1\left(\frac{\mu(x\rho(t) + t) + x\rho(t)}{\lambda(t)}, t\right)\right)}{\partial p_1\left(\frac{\mu(x\rho(t) + t) + x\rho(t)}{\lambda(t)}, t\right)} + c_1\right) \\ & \times \left(f_1\left(p_1\left(\frac{\mu(x\rho(t) + t) + x\rho(t)}{\lambda(t)}, t\right)\right) + p_2(x,t)\right) \\ & \times R(t + x(\phi_2^{-1}(t) - t)) \end{aligned} \tag{132}$$

where

$$\rho(t) = \phi_2^{-1}(t) - t \tag{133}$$

transforms the system (83) and (84) to the "target system" with the control law given by Eq. (92). The target system is given by

$$\dot{Z}_1(t) = -c_1 Z_1(t) + Z_2(\phi_1(t)) \tag{134}$$

$$\dot{Z}_2(t) = -c_2 Z_2(t) + W(\phi_2(t)) \tag{135}$$

where

$$W(\theta) = 0, \quad \theta \geq 0 \tag{136}$$

Proof. Before we start our recursive procedure we rewrite the target system using transport PDEs as

$$\dot{Z}_1(t) = -c_1 Z_1(t) + \zeta_2(0,t) \tag{137}$$

$$\zeta_{2_t}(x,t) = \pi_1(x,t)\zeta_{2_x}(x,t) \tag{138}$$

$$\zeta_2(1,t) = Z_2(t) \tag{139}$$

$$\dot{Z}_2(t) = -c_2 Z_2(t) + w(0,t) \tag{140}$$

$$w_t(x,t) = \pi_2(x,t) w_x(x,t) \tag{141}$$

$$w(1,t) = 0. \tag{142}$$

Note that

$$\zeta_2(x,t) = Z_2\left(\phi_1\left(t + x\left(\phi_1^{-1}(t) - t\right)\right)\right), \quad x \in [0,1] \tag{143}$$

Define

$$\zeta_2(x,t) = \xi_2(x,t) + f_1\left(p_1\left(\frac{\mu(t)}{\lambda(t)}x, t\right)\right) + c_1 p_1\left(\frac{\mu(t)}{\lambda(t)}x, t\right) \tag{144}$$

Then using Eqs. (105) and (115) we get

$$\dot{X}_1(t) = -c_1 X_1(t) + \zeta_2(0,t) \tag{145}$$

Using relation (117) we have that

$$p_{1_t}(f(x,t),t) = \pi_3(f(x,t),t) p_{1_f}(f(x,t),t) \tag{146}$$

with

$$f(x,t) = \frac{\mu(t)}{\lambda(t)}x \tag{147}$$

From Eq. (144) and by using relation (146) together with Eq. (106) we get

$$\begin{aligned}\zeta_{2_t}(x,t) &= \pi_1(x,t)\xi_{2_x}(x,t) \\ &+ \left(\frac{\partial f_1(p_1(f(x,t),t))}{\partial p_1(f(x,t),t)} + c_1\right) p_{1_f}(f(x,t),t)(f_t(x,t) \\ &+ \pi_3(f(x,t),t))\end{aligned} \tag{148}$$

and

$$\begin{aligned}\pi_1(x,t)\zeta_{2_x}(x,t) = \pi_1(x,t)\xi_{2_x}(x,t) + \left(\frac{\partial f_1(p_1(f(x,t),t))}{\partial p_1(f(x,t),t)} + c_1\right) \\ \times p_{1_f}(f(x,t),t)\pi_1(x,t) f_x(x,t)\end{aligned} \tag{149}$$

Using Eqs. (119), (130), and (131) and the definition of $f(x,t)$ in Eq. (147) we have that

$$\begin{aligned}f_t(x,t) + \pi_3(f(x,t),t) - \pi_1(x,t) f_x(x,t) \\ = \frac{1+\mu'(t)x}{\lambda(t)} - \frac{1+x\mu'(t)}{\mu(t)}\frac{\mu(t)}{\lambda(t)} = 0\end{aligned} \tag{150}$$

Consequently, Eq. (138) holds. From Eq. (139) and by using Eq. (138) we have that

$$\begin{aligned}\dot{Z}_2(t) &= f_2(X_1(t), X_2(t)) + u(0,t) \\ &+ \left(\frac{\partial f_1(p_1(f(1,t),t))}{\partial p_1(f(1,t),t)} + c_1\right) p_{1_f}(f(1,t),t)\pi_1(1,t) f_x(1,t)\end{aligned} \tag{151}$$

With the help of Eqs. (111), (113), (130), (131), and (147) we rewrite the previous relation as

$$\begin{aligned}\dot{Z}_2(t) &= f_2(X_1(t),X_2(t)) + u(0,t) \\ &+ \left(\frac{\partial f_1(p_1(f(1,t),t))}{\partial p_1(f(1,t),t)} + c_1\right)\left(f_1(p_1(f(1,t),t)) + X_2(t)\right)\frac{d\phi_1^{-1}(t)}{dt}\end{aligned} \tag{152}$$

Define now for notational convenience

$$g_1(x,t) = \frac{\rho(t)}{\lambda(t)}x \tag{153}$$

$$g_2(x,t) = \frac{\mu(x\rho(t)+t) + x\rho(t)}{\lambda(t)} \tag{154}$$

By noting that $g_1(x,t)$, $g_2(x,t)$, and $R(t + x(\phi_2^{-1}(t) - t))$ satisfy the following boundary value problems

$$g_{1_t}(x,t) + \pi_3(g_1(x,t),t) = \pi_2(x,t) g_{1_x}(x,t) \tag{155}$$

$$g_1(0,t) = 0 \tag{156}$$

$$g_{2_t}(x,t) + \pi_3(g_2(x,t),t) = \pi_2(x,t) g_{2_x}(x,t) \tag{157}$$

$$g_2(0,t) = \frac{\mu(t)}{\lambda(t)} \tag{158}$$

$$R_t\left(t + x\left(\phi_2^{-1}(t) - t\right)\right) = \pi_2(x,t) R_x\left(t + x\left(\phi_2^{-1}(t) - t\right)\right) \tag{159}$$

and using Eq. (132) we get Eq. (141). Substituting Eq. (132) for $x = 0$ into Eq. (152) we get Eq. (140). Using the facts that

$$\begin{aligned}g_2(1,t) &= \mu(\rho(t)+t) + \rho(t)/\lambda(t) \\ &= \mu\left(\phi_2^{-1}(t)\right) + \phi_2^{-1}(t) - t/\phi_1^{-1} \circ \phi_2^{-1}(t) - t \\ &= \phi_1^{-1} \circ \phi_2^{-1}(t) - t/\phi_1^{-1} \circ \phi_2^{-1}(t) - t = 1\end{aligned}$$

and $p_1(\rho(t)/\lambda(t),t) = P_1(\phi_1(t))$, with the controller (92) we get Eq. (142). Assuming an initial condition for (132) as $w(x,0) = w_0(x)$, and by defining a new variable W as $w_0(x) = W(\phi_2(x(\phi_2^{-1}(0))))$ for all $x \in [0,1]$, we get that

$$w(x,t) = \left\{\begin{array}{cc} W\left(\phi_2\left(t + x\left(\phi_2^{-1}(t) - t\right)\right)\right), & 0 \leq t + x\left(\phi_2^{-1}(t) - t\right) \leq \phi_2^{-1}(0) \\ 0, & t + x\left(\phi_2^{-1}(t) - t\right) \geq \phi_2^{-1}(0) \end{array}\right\} \tag{160}$$

Defining $\theta = t + x(\phi_2^{-1}(t) - t)$ one gets Eq. (136).

The functions $f(x,t)$, $g_1(x,t)$, and $g_2(x,t)$ are increasing with respect to x and they take values within [0,1]. Hence, whenever they appear as the first argument in p_1 or p_2 they guarantee that the first argument of p_1 or p_2 varies within the interval [0,1] which is consistent with the definitions (103) and (104). We now state the inverse of the transformation (128)–(132).

Lemma 9. *The inverse transformation of Eqs.* (128)–(132) *is defined as*

$$X_1(t) = Z_1(t) \tag{161}$$

$$X_2(t) = Z_2(t) - \left(f_1\left(\eta_1\left(\frac{\mu(t)}{\lambda(t)}, t\right)\right) + c_1\eta_1\left(\frac{\mu(t)}{\lambda(t)}, t\right)\right) \tag{162}$$

$$\begin{aligned} u(x,t) = {} & w(x,t) - f_2\left(\eta_1\left(\frac{\rho(t)}{\lambda(t)}x, t\right), \eta_2(x,t)\right) \\ & - f_1\left(\eta_1\left(\frac{\mu(x\rho(t)+t) + x\rho(t)}{\lambda(t)}, t\right)\right) \\ & - c_1\eta_1\left(\frac{\mu(x\rho(t)+t) + x\rho(t)}{\lambda(t)}, t\right)\right) - c_2\eta_2(x,t) \\ & - \left(\frac{\partial f_1\left(\eta_1\left(\frac{\mu(x\rho(t)+t) + x\rho(t)}{\lambda(t)}, t\right)\right)}{\partial\eta_1\left(\frac{\mu(x\rho(t)+t) + x\rho(t)}{\lambda(t)}, t\right)} + c_1\right) \\ & \times \left(-c_1\eta_1\left(\frac{\mu(x\rho(t)+t) + x\rho(t)}{\lambda(t)}, t\right) + \eta_2(x,t)\right) \\ & \times R\left(t + x(\phi_2^{-1}(t) - t)\right) \end{aligned} \tag{163}$$

where $\eta_1(x,t)$ and $\eta_2(x,t)$ are the predictors of the transformed states $Z_1(t)$ and $Z_2(t)$ and hence they satisfy

$$\begin{aligned} \eta_1(x,t) = {} & Z_1(t) + \left(\psi^{-1}(t) - t\right) \\ & \times \int_0^x \left(-c_1\eta_1(y,t) + \eta_2\left(\frac{\phi_1\left(t + (\psi^{-1}(t) - t)x\right) - t}{\phi_2^{-1}(t) - t}, t\right)\right) dy \end{aligned} \tag{164}$$

$$\eta_2(x,t) = Z_2(t) + \left(\phi_2^{-1}(t) - t\right)\int_0^x \left(-c_2\eta_2(y,t) + w(y,t)\right) dy \tag{165}$$

where x varies *in* [0,1].

Proof. Using Eq. (144) we have that

$$\xi_2(x,t) = \zeta_2(x,t) - f_1\left(\eta_1\left(\frac{\mu(t)}{\lambda(t)}x, t\right)\right) + c_1\eta_1\left(\frac{\mu(t)}{\lambda(t)}x, t\right) \tag{166}$$

With Eq. (161) we get Eq. (105). Consider further

$$\begin{aligned} \dot{X}_2(t) = {} & -c_2Z_2(t) + w(0,t) \\ & - \left(\frac{\partial f_1(\eta_1(f(1,t),t))}{\partial\eta_1(f(1,t),t)} + c_1\right)\left(-c_1\eta_1\left(\frac{\mu(t)}{\lambda(t)}, t\right) + Z_2(t)\right)R(t) \end{aligned} \tag{167}$$

Then, with Eq. (163) for $x = 0$ we get Eq. (108).

We now prove stability of the "target system."

Lemma 10. *The target system Eq.* (134)–(136) *is globally asymptotically stable in the sense that there exist positive constants G and g such that*

$$\begin{aligned} & |Z_1(t)| + \|Z_2(t)\|_\infty + \|W(t)\|_\infty \le G(|Z_1(0)| \\ & \quad + \|Z_2(0)\|_\infty + \|W(0)\|_\infty)e^{-gt} \end{aligned} \tag{168}$$

Proof. Solving explicitly (134)–(135) and using Eq. (136) we have for all $t \ge 0$ that

$$\begin{aligned} |Z_1(t)| \le {} & \left(\Gamma_z + \frac{\Gamma_w}{\sqrt{c_1|c_1\pi^*_{1_{\phi_1}} - 2c_2|\inf_{\theta \ge \phi_1^{-1}(0)}\phi_1'(\theta)}}\right) \\ & \times \left(e^{-c_1\min\left\{1, \frac{\pi^*_{1_{\phi_1}}}{2}\right\}t} + e^{-c_2\max\{\phi_1(t),0\}}\right) \end{aligned} \tag{169}$$

$$\|Z_2(t)\|_\infty \le 2\Gamma_w e^{-c_2\max\{\phi_1(t),0\}} \tag{170}$$

where

$$\Gamma_z = |Z_1(0)| + \phi_1^{-1}(0)e^{c_1\phi_1^{-1}(0)}\|Z_2(0)\|_\infty \tag{171}$$

$$\Gamma_w = |Z_2(0)| + \phi_2^{-1}(0)e^{c_2\phi_2^{-1}(0)}\|W(0)\|_\infty \tag{172}$$

With the help of Eqs. (45), (46) and Eqs. (53), (54) we have

$$\|W(t)\|_\infty \le e^{-c\pi^*_{0_{\phi_2}}\min\left\{1, \pi^*_{1_{\phi_2}}\right\}t} e^c\|W(0)\|_\infty \tag{173}$$

Combining Eqs. (169)–(173) and using the facts that $\max\{\phi_1(t), 0\} \ge t - D_1(t)$ and that

$$\begin{aligned} \sup_{t\ge0} D_1(t) & = \sup_{t\ge0} t - \phi_1(t) \\ & \le \sup_{0\le t\le\phi_1^{-1}(0)} t - \phi_1(t) + \sup_{t\ge\phi_1^{-1}(0)} t - \phi_1(t) \end{aligned} \tag{174}$$

the proof is complete with

$$\begin{aligned} G = {} & 3\left(1 + b + e^c + \phi_1^{-1}(0)e^{c_1\phi_1^{-1}(0)} + b\phi_2^{-1}(0)e^{c_2\phi_2^{-1}(0)}\right) \\ & \times e^{g\left(\phi_1^{-1}(0) - \phi_1(0) + \frac{1}{\pi^*_{0_{\phi_1}}}\right)} \end{aligned} \tag{175}$$

$$b = 2 + \frac{1}{\sqrt{c_1|c_1\pi^*_{1_{\phi_1}} - 2c_2|\inf_{\theta\ge\phi_1^{-1}(0)}\phi_1'(\theta)}} \tag{176}$$

$$g = \min\left\{c_1, \frac{c_1\pi^*_{1_{\phi_1}}}{2}, c_2, c\pi^*_{0_{\phi_2}}\min\left\{1, \pi^*_{1_{\phi_2}}\right\}\right\} \tag{177}$$

We have to relate now the stability of the "target system" with the stability of the system in the original variables.

Lemma 11. *There exists a class $\mathcal{K}_\infty$ function $\hat{\alpha}_1$ such that the following holds for all $t \ge 0$*

$$\begin{aligned} & |p_1(x,t)| + |p_2(x,t)| \le \hat{\alpha}_1\left(|X_1(t)| + \|X_2(t)\|_\infty + \|U(t)\|_\infty\right) \\ & \forall x \in [0,1] \end{aligned} \tag{178}$$

Proof. By performing a change of variables as $x = y/(\psi^{-1}(t) - t)$ in Eq. (113) and $x = y/(\phi_2^{-1}(t) - t)$ in Eq. (114) we re-write the ODE (in x) system (113)–(115) as

$$p'_{1_y}(y,t) = f_1\left(p'_1(y,t)\right) + p'_2(\phi_1(t+y) - t, t), \quad y \in [0, \psi^{-1}(t) - t] \tag{179}$$

$$p'_{2_y}(y,t) = f_2\left(p'_1(y,t), p'_2(y,t)\right) + u'(y,t), \quad y \in [0, \phi_2^{-1}(t) - t] \tag{180}$$

where

$$p'_1(y,t) = p_1\left(\frac{y}{\psi^{-1}(t) - t}, t\right), \quad y \in [0, \psi^{-1}(t) - t] \tag{181}$$

$$p'_2(y,t) = p_2\left(\frac{y}{\phi_2^{-1}(t) - t}, t\right), \quad y \in [0, \phi_2^{-1}(t) - t] \tag{182}$$

$$u'(y,t) = u\left(\frac{y}{\phi_2^{-1}(t) - t}, t\right), \quad y \in [0, \phi_2^{-1}(t) - t] \tag{183}$$

Note that we view the function $\phi_1(t+y) - t$ where y is the independent variable as the function $\phi_1(y)$ in Eq. (83) (note that $\phi_1(t+y) - t$ satisfies both assumptions 3–4 for all $t \geq 0$ and $y \in \left[0, \psi^{-1}(t) - t\right]$). Under assumption 5 and using [43] together with the fact that $f_1(0) = f_2(0,0) = 0$ we conclude that there exist a class $\mathcal{K}$ function μ and a class $\mathcal{K}_\infty$ function ν such that for all $t \geq 0$ the following holds

$$|X_1(t)| + \sup_{\phi_1(t) \leq \theta \leq t} |X_2(\theta)| \leq \mu(t)\nu \times \left(|X_1(0)| + \sup_{\phi_1(0) \leq \theta \leq 0} |X_2(\theta)| + \sup_{\theta \in [0,t]} |U(\phi_2(\theta))|\right) \tag{184}$$

Comparing the ODE (in y) system (179) and (180) with Eq. (83) and (84) we get for all $y \in [0, \phi_2^{-1}(t) - t]$ that

$$|p'_1(y,t)| + |p'_2(y,t)| \leq \mu\left(\frac{1}{\pi^*_{0_{\phi_2}}}\right)\nu \times \left(|p'_1(0,t)| + \sup_{\phi_1(t) - t \leq \theta \leq 0} |p'_2(\theta,t)| + \sup_{\theta \in [0,y]} |u'(\theta,t)|\right) \tag{185}$$

Using Eqs. (99) and (120),(121) we get for all $y \in [0, \phi_2^{-1}(t) - t]$

$$|p'_1(y,t)| + |p'_2(y,t)| \leq \mu\left(\frac{1}{\pi^*_{0_{\phi_2}}}\right)\nu\left(|X_1(t)| + \|X_2(t)\|_\infty + \|U(t)\|_\infty\right) \tag{186}$$

Define now the following function:

$$\omega(Y,r) = \sup\Bigg\{|p'_1(h)| : \phi_2^{-1}(t) - t \leq h \leq Y \leq \psi^{-1}(t) - t, \quad |p'_1\left(\phi_2^{-1}(t) - t\right)| + \sup_{\phi_2^{-1}(t) - t \leq \theta \leq \psi^{-1}(t) - t} |p'_2(\phi_1(t+\theta) - t)| \leq r$$
$$\text{where } p'_1(y) \text{ and } p'_2(\phi_1(t+y) - t) \text{ satisfy Eqs. (179) and (180) for all } \phi_2^{-1}(t) - t \leq y \leq \psi^{-1}(t) - t\Bigg\} \tag{187}$$

From the forward-completeness assumption we know that $\omega(Y, r)$ is finite. Moreover, one concludes that for all $\phi_2^{-1}(t) - t \leq y \leq \psi^{-1}(t) - t$ the following holds

$$|p'_1(y)| \leq \omega\left(y, |p'_1\left(\phi_2^{-1}(t) - t\right)| + \sup_{\phi_2^{-1}(t) - t \leq \theta \leq \psi^{-1}(t) - t} |p'_2(\phi_1(t+\theta) - t)|\right) \tag{188}$$

Since from the definition of ω, we conclude that for each fixed $\phi_2^{-1}(t) - t \leq y \leq \psi^{-1}(t) - t$ the mapping $\omega(y, \cdot)$ is increasing and for each fixed r the mapping $\omega(\cdot, r)$ is increasing we get for all $\phi_2^{-1}(t) - t \leq y \leq \psi^{-1}(t) - t$ that

$$|p'_1(y)| \leq \omega\left(\frac{1}{\pi^*_{0_\psi}}, |p'_1\left(\phi_2^{-1}(t) - t\right)| + \sup_{\phi_2^{-1}(t) - t \leq \theta \leq \psi^{-1}(t) - t} |p'_2(\phi_1(t+\theta) - t)|\right) \tag{189}$$

Using the fact that $f_1(0) = 0$ we conclude that $\omega(y, 0) = 0$ for all y and hence, there exists a class $\mathcal{K}_\infty$ function α^* such that

$$|p'_1(y)| \leq \alpha^*\left(|p'_1\left(\phi_2^{-1}(t) - t\right)| + \sup_{\phi_2^{-1}(t) - t \leq \theta \leq \psi^{-1}(t) - t} |p'_2(\phi_1(t+\theta) - t)|\right) \tag{190}$$

where we absorb the finite constant $\pi^*_{0_\psi}$ into α^*. Therefore, using the fact that $\phi_1(t) - t \leq \phi_1\left(\phi_2^{-1}(t)\right) - t \leq \phi_1(\theta + t) - t \leq \phi_2^{-1}(t) - t$ for all $\phi_2^{-1}(t) - t \leq \theta \leq \psi^{-1}(t) - t$ and (186) the proof is complete.

Lemma 12. *There exists a class $\mathcal{K}_\infty$ function $\hat{\alpha}_4$ such that the following holds for all $t \geq 0$*

$$|\eta_1(x,t)| + |\eta_2(x,t)| \leq \hat{\alpha}_4\left(|Z_1(t)| + \|Z_2(t)\|_\infty + \|W(t)\|_\infty\right) \quad \forall x \in [0,1]. \tag{191}$$

Proof. Relations (164)–(165) can be solved explicitly as

$$\eta_1(x,t) = Z_1(t)e^{-c_1\left(\psi^{-1}(t) - t\right)x} + \left(\psi^{-1}(t) - t\right)\int_0^x e^{-c_1\left(\psi^{-1}(t) - t\right)(x-y)}\eta_2 \times \left(\frac{\phi_1\left(t + \left(\psi^{-1}(t) - t\right)y\right) - t}{\phi_2^{-1}(t) - t}, t\right)dy \tag{192}$$

$$\eta_2(x,t) = Z_2(t)e^{-c_1\left(\phi_2^{-1}(t) - t\right)x} + \left(\phi_2^{-1}(t) - t\right)\int_0^x e^{-c_2\left(\phi_2^{-1}(t) - t\right)(x-y)}w(y,t)dy \tag{193}$$

From relation (193) and using the fact that $0 \leq \phi_2^{-1}(t) - t \leq 1/\pi^*_{0_{\phi_2}}$ we get

$$|\eta_2(x,t)| \leq |Z_2(t)| + \frac{1}{\pi^*_{0_{\phi_2}}}\sup_{x \in [0,1]} |w(x,t)|, \quad \forall x \in [0,1] \tag{194}$$

By changing variables in the integral in Eq. (192) and by using the fact that $\eta_2(x,t) = Z_2\left(t + x\left(\phi_2^{-1}(t) - t\right)\right)$ we write

$$|\eta_1(x,t)| \leq |Z_1(t)| + \int_0^{\phi_1^{-1}(t) - t} |Z_2(\phi_1(t+y))|dy + \int_{\inf_{t \geq 0}\left\{\phi_1^{-1}(t) - t\right\}}^{\psi^{-1}(t) - t} \left|\eta_2\left(\frac{\phi_1(t+y) - t}{\phi_2^{-1}(t) - t}, t\right)\right|dy \tag{195}$$

Using Eq. (194) and the uniform boundness of the functions $\phi_1^{-1}(t) - t$ and $\psi^{-1}(t) - t$ for all $t \geq 0$ the proof of the lemma is complete.

Proof of Theorem 2: Since $f_1(X_1), f_2(X_1, X_2)$ and $(\partial f_1(X_1))/\partial X_1$ are continuous, there exist class $\mathcal{K}_\infty$ functions ρ_1, ρ_2 and $\hat{\delta}$ such that

$$f_1(X_1) \leq \rho_1(|X_1|) \tag{196}$$

$$f_2(X_1, X_2) \leq \rho_2(|X_1| + |X_2|) \tag{197}$$

$$\frac{\partial f_1(X_1)}{\partial X_1} \leq \left. \frac{\partial f_1(X_1)}{\partial X_1} \right|_{X_1=0} + \hat{\delta}(|X_1|) \tag{198}$$

Using relations (128),(129), (132) and with the help of Lemma 11, we have

$$|Z_1(t)| = |X_1(t)| \tag{199}$$

$$\|Z_2(t)\|_\infty \leq \rho_3\left(|X_1(t)| + \|X_2(t)\|_\infty + \|U(t)\|_\infty\right) \tag{200}$$

$$\|W(t)\|_\infty \leq \rho_4\left(|X_1(t)| + \|X_2(t)\|_\infty + \|U(t)\|_\infty\right) \tag{201}$$

where class $\mathcal{K}_\infty$ functions ρ_3 and ρ_4 are

$$\rho_3(s) = s + \rho_1^\circ \hat{\alpha}_1(s) + c_1 \hat{\alpha}_1(s) \tag{202}$$

$$\begin{aligned} \rho_4(s) = {} & \rho_2^\circ \hat{\alpha}_1(s) + c_2\left((1 + c_1)\hat{\alpha}_1(s) + \rho_1^\circ \hat{\alpha}_1(s)\right) \\ & + \left(\hat{\delta}^\circ \hat{\alpha}_1(s) + c_1\right)\left(\rho_1^\circ \hat{\alpha}_1(s) + \hat{\alpha}_1(s)\right) \frac{1}{\inf_{\theta \geq \phi^{-1}(0)} \phi_1'(\theta)} \end{aligned} \tag{203}$$

Hence

$$\begin{aligned} |Z_1(t)| + \|Z_2(t)\|_\infty + \|W(t)\|_\infty \leq \rho_5(|X_1(t)| \\ + \|X_2(t)\|_\infty + \|U(t)\|_\infty) \end{aligned} \tag{204}$$

with

$$\rho_5(s) = s + \rho_3(s) + \rho_4(s) \tag{205}$$

Similarly, using relations (161)–(163) and with the help of Lemma 12 we have for some class $\mathcal{K}_\infty$ function ρ_6 that

$$\begin{aligned} |X_1(t)| + \|X_2(t)\|_\infty + \|U(t)\|_\infty \leq \rho_6(|X_1(t)| \\ + \|Z_2(t)\|_\infty + \|W(t)\|_\infty) \end{aligned} \tag{206}$$

Using Eqs. (204)–(206) and with the help of Lemma 10 the theorem is proved with

$$\begin{aligned} |X_1(t)| + \|X_2(t)\|_\infty + \|U(t)\|_\infty \leq \rho_6(G\rho_5(|X_1(0)| \\ + \|X_2(0)\|_\infty + \|U(0)\|_\infty)e^{-gt}) \end{aligned} \tag{207}$$

5 Examples

Example 1. We consider a special case of system (1)

$$\dot{X}_1(t) = X_2(t) - X_2(t)^2 U(\phi(t)) \tag{208}$$

$$\dot{X}_2(t) = U(\phi(t)) \tag{209}$$

System (208) and (209) is forward complete since it is in the strict-feedforward form. Moreover, a time-invariant controller that renders the closed loop system input-to-state stable is

$$U(t) = -X_1(t) - 2X_2(t) - \frac{1}{3} X_2(t)^3 \tag{210}$$

Assume now a form for the function $\phi(t)$ as

$$\phi(t) = t - \frac{1 + t}{1 + 2t} \tag{211}$$

Consequently

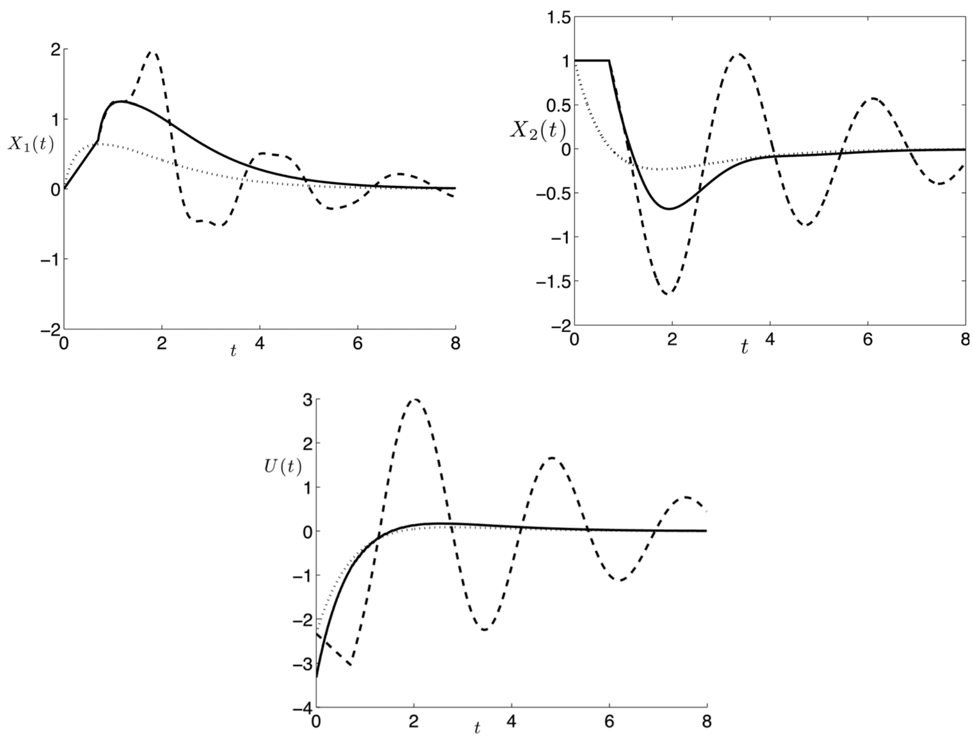

Fig. 1 System's response for Example 1. Dot-lines: System with $\phi(t) = t$ and the controller (210). Dash-lines: system with $\phi(t)$ as in Eq. (211) and the uncompensated controller (210). Solid-lines: System with $\phi(t)$ as in Eq. (211) and the delay-compensating controller (214).

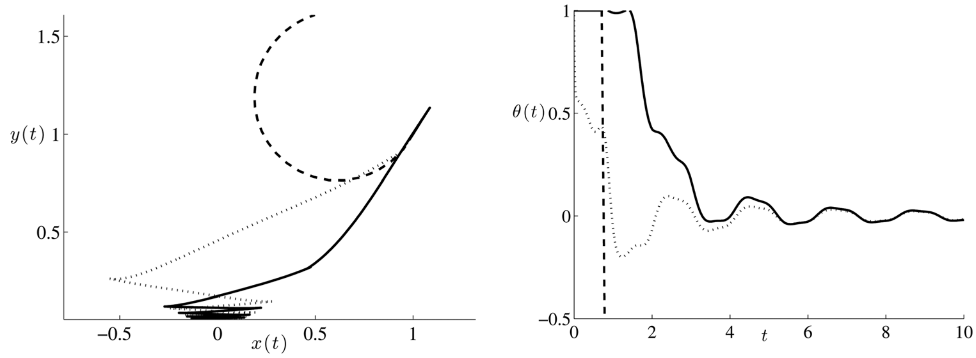

Fig. 2 The trajectory of the robot and the heading $\theta(t)$ with the compensated controller (222)–(225), (227)–(229) (solid line), the uncompensated controller (222)–(225), (226) (dashed line) and the controller (222)–(225), (226) for the delay-free system (dotted line) with initial conditions $x(0) = y(0) = \theta(0) = 1$ and $\omega(s) = v(s) = 0$ for all $\phi(0) \le s \le 0$

$$\phi'(t) = 1 + \frac{1}{(1+2t)^2} \tag{212}$$

$$\phi^{-1}(t) = t + \frac{t+1}{\sqrt{(t+1)^2+1}+t} \tag{213}$$

From expressions (211)–(213) one can see that the function $\phi(t)$ in Eq. (211) satisfies both assumptions 3 and 4. The controller that compensates the time-varying delay is given by

$$U(t) = -P_1(t) - 2P_2(t) - \frac{1}{3}P_2(t)^3 \tag{214}$$

where

$$\begin{aligned} P_1(t) &= \left(\phi^{-1}(t)-t\right)\int_0^1 \left(P_2\left(\phi\left(t+y\left(\phi^{-1}(t)-t\right)\right)\right)\right. \\ &\quad - P_2\left(\phi\left(t+y\left(\phi^{-1}(t)-t\right)\right)\right)^2 \\ &\quad \times U\left(\phi\left(t+y\left(\phi^{-1}(t)-t\right)\right)\right)\Big)dy + X_1(t) \\ &= \int_{\phi(t)}^t \left(P_2(\theta) - P_2(\theta)^2 U(\theta)\right)\frac{d\theta}{\phi'\left(\phi^{-1}(\theta)\right)} + X_1(t) \end{aligned} \tag{215}$$

$$\begin{aligned} P_2(t) &= \left(\phi^{-1}(t)-t\right)\int_0^1 U\left(\phi\left(t+y\left(\phi^{-1}(t)-t\right)\right)\right)dy + X_2(t) \\ &= \int_{\phi(t)}^t U(\theta)\frac{d\theta}{\phi'\left(\phi^{-1}(\theta)\right)} + X_2(t) \end{aligned} \tag{216}$$

and $\phi(t)$ and $\phi^{-1}(t)$ are given in Eqs. (211) and (213), respectively. The response of the system for initial conditions $X_1(0)=0$, $X_2(0)=1$, and $U(\theta)=0$ for all $\theta \in [\phi(0), 0]$ is shown in Fig. 1. We note here that in the case of a constant delay, i.e., in the case where $\phi(t) = t - D$, and hence, $\phi'(t) = 1$, the control law (214) remains the same, whereas the predictor states are defined as

$$P_1(t) = \int_{t-D}^t \left(P_2(\theta) - P_2(\theta)^2 U(\theta)\right)d\theta + X_1(t) \tag{217}$$

$$P_2(t) = \int_{t-D}^t U(\theta)d\theta + X_2(t) \tag{218}$$

Example 2. We consider the problem of stabilizing a mobile robot modeled as

$$\dot{x}(t) = v(\phi(t))\cos(\theta(t)) \tag{219}$$

$$\dot{y}(t) = v(\phi(t))\sin(\theta(t)) \tag{220}$$

$$\dot{\theta}(t) = \omega(\phi(t)) \tag{221}$$

subject to the input delay given by Eq. (211), where $(x(t), y(t))$ is position of the robot, $\theta(t)$ is heading, $v(t)$ is speed and $\omega(t)$ is turning rate. When $D = 0$ (i.e., $\phi(t) = t$) a time-varying stabilizing controller for this system is proposed in Ref. [44] as

$$\begin{aligned} \omega(t) = &-5P(t)^2\cos\left(3\phi^{-1}(t)\right) \\ &- P(t)Q(t)\left(1+25\cos^2\left(3\phi^{-1}(t)\right)\right) - \Theta(t) \end{aligned} \tag{222}$$

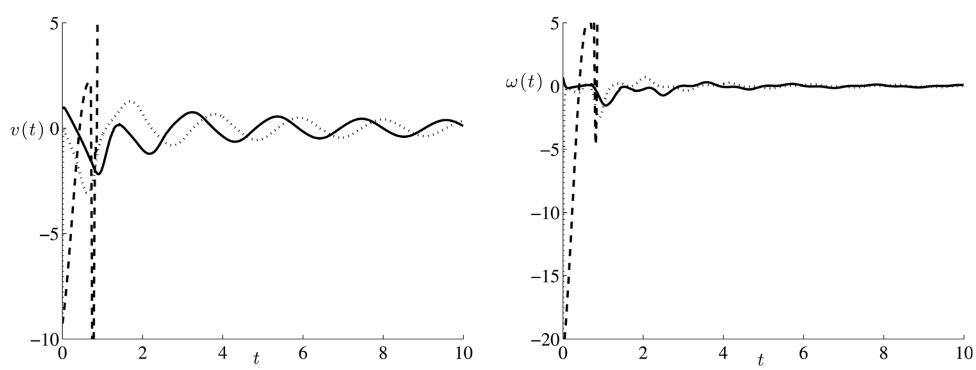

Fig. 3 The control efforts $v(t)$ and $\omega(t)$ with the controller (222)–(225), (227)–(229) (solid line), the controller (222)–(225), (226) (dashed line) and the controller (222)–(225), (226) for the delay-free system (dotted line) with initial conditions $x(0) = y(0) = \theta(0) = 1$ and $\omega(s) = v(s) = 0$ for all $\phi(0) \le s \le 0$

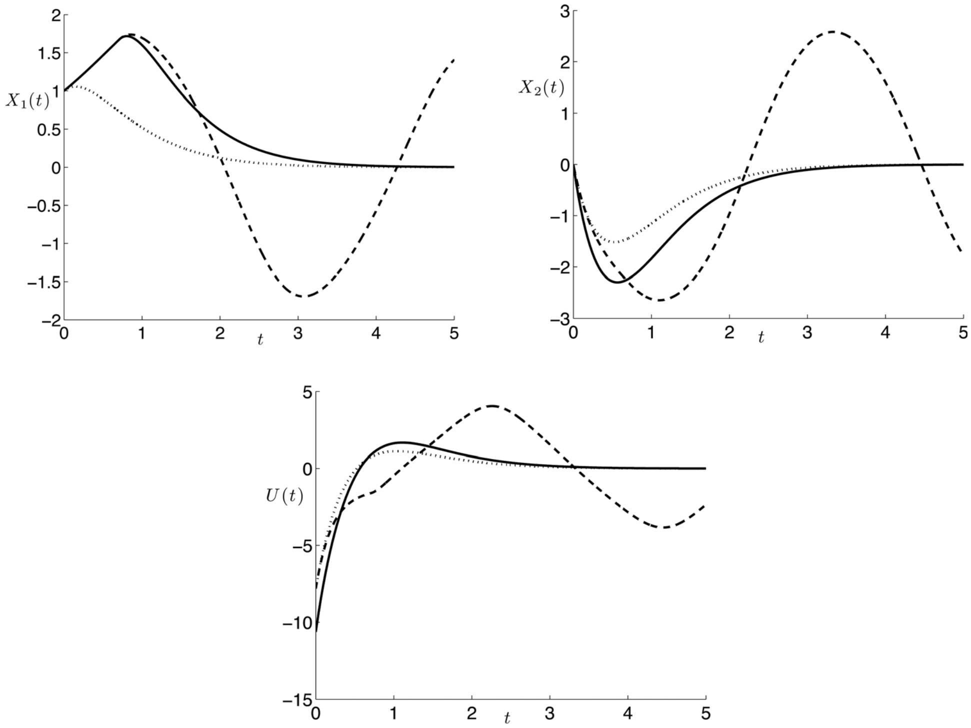

Fig. 4 System's response for Example 3. Dot-lines: System with $\varphi(t) = t$ and the uncompensated controller. Dash-lines: System with $\phi(t)$ as in Eq. (211) and the uncompensated controller. Solid-lines: System with $\phi(t)$ as in Eq. (211) and the delay-compensating controller.

$$v(t) = -P(t) + 5Q(t)\left(\sin\left(3\phi^{-1}(t)\right) - \cos\left(3\phi^{-1}(t)\right)\right) + Q(t)\omega(t) \tag{223}$$

$$P(t) = X(t)\cos(\Theta(t)) + Y(t)\sin(\Theta(t)) \tag{224}$$

$$Q(t) = X(t)\sin(\Theta(t)) - Y(t)\cos(\Theta(t)) \tag{225}$$

with

$$X = x, \quad Y = y, \quad \Theta = \theta, \quad \phi^{-1}(t) = t \tag{226}$$

The predictor-based version of Eqs. (222)–(225) is given with

$$X(t) = x(t) + \int_{\phi(t)}^{t} \frac{v(s)\cos(\Theta(s))ds}{\phi'\left(\phi^{-1}(s)\right)} \tag{227}$$

$$Y(t) = y(t) + \int_{\phi(t)}^{t} \frac{v(s)\sin(\Theta(s))ds}{\phi'\left(\phi^{-1}(s)\right)} \tag{228}$$

$$\Theta(t) = \theta(t) + \int_{\phi(t)}^{t} \frac{\omega(s)ds}{\phi'\left(\phi^{-1}(s)\right)} \tag{229}$$

The initial conditions are chosen as $x(0) = y(0) = \theta\ (0) = 1$ and $\omega(s) = v(s) = 0$ for all $\phi(0) \le s \le 0$. From the given initial conditions one can verify that the controller "kicks in" at the time instant $t_0 = \phi^{-1}(0) = 1/\sqrt{2}$. In Fig. 2 we show the trajectory of the robot in the xy plane and the heading, whereas in Fig. 3 we show the response of the controls $v(t)$ and $\omega(t)$. In the case of the uncompensated controller (222)–(225), (226), the system is unstable.

Example 3. In this example we consider the following system

$$\dot{X}_1(t) = \sin(X_1(t)) + X_2(\phi(t)) \tag{230}$$

$$\dot{X}_2(t) = U(t) \tag{231}$$

where the function $\phi(t)$ is the function of Example 1. We choose the initial conditions of the plant as $X_1(0) = 1$ and $X_2(s) = 0$ for all $s \in [\phi(0), 0]$. The controller for this system is

$$\begin{aligned} U(t) = &- c_2(X_2(t) + c_1 P_1(t) + \sin(P_1(t))) \\ &- (c_1 + \cos(P_1(t)))(\sin(P_1(t)) + X_2(t))\frac{d\phi^{-1}(t)}{dt} \end{aligned} \tag{232}$$

where we choose $c_1 = c_2 = 2$ and

$$P_1(t) = X_1(t) + \int_{\phi(t)}^{t} (\sin(P_1(\theta)) + X_2(\theta))\frac{d\theta}{\phi'\left(\phi^{-1}(\theta)\right)} \tag{233}$$

In Figs. 2 and 4 we show the response of the system in comparison with the uncompensated controller, i.e., the backstepping controller (232) which assumes $\phi(t) = t$.

6 Conclusions

We introduce a design methodology for nonlinear systems with time-varying input and state delays. For systems with input delay we achieve asymptotic stability based on a time-varying, infinite-dimensional backstepping transformation of the actuator state. For the case of simultaneous input and state delays we employ time-varying, infinite-dimensional backstepping transformations both in the state of the system and the actuator state. In both cases, we prove stability in the original variables using the fact that the backstepping transformations are invertible and using a Lyapunov function that we construct. Our numerical example illustrates the effectiveness of the control law.

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