Compensation of State-Dependent Input Delay for Nonlinear Systems

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***Abstract*—We introduce and solve stabilization problems for linear and nonlinear systems with state-dependent input delay. Since the state dependence of the delay makes the prediction horizon dependent on the future value of the state, which means that it is impossible to know *a priori* how far in the future the prediction is needed, the key design challenge is how to determine the predictor state. We resolve this challenge and establish closed-loop stability of the resulting infinite-dimensional nonlinear system for general non-negative-valued delay functions of the state. Due to an inherent limitation on the allowable delay rate in stabilization of systems with time-varying input delays, in the case of state-dependent delay, where the delay rate becomes dependent on the gradient of the delay function and on the state and control input, only regional stability results are achievable. For forward-complete systems, we establish an estimate of the region of attraction in the state space of the infinite-dimensional closed-loop nonlinear system and for linear systems we prove exponential stability. Global stability is established under a restrictive Lyapunov-like condition, which has to be *a priori* verified, that the delay rate be bounded by unity, irrespective of the values of the state and input. We also establish local asymptotic stability for locally stabilizable systems in the absence of the delay. Several illustrative examples are provided, including unicycle stabilization subject to input delay that grows with the distance from the reference position.**

***Index Terms*—Delay systems, feedback, nonlinear control systems.**

I. Introduction

A. Motivation

STATE-DEPENDENT delays are ubiquitous. For example, in control over networks, it makes sense to send control signals less frequently when the state is small and more frequently when the state is large [11]. Network congestion control typically ignores the dependency of the round-trip time (RTT) from the queue length, which can lead to the instability of the underlying network [28]. In the control of mobile robots, the magnitude of the delay depends on the distance of the robot from the operator interface [42]. *A priori* known functions of time are employed to model state-dependent delays in transmission channels of communication networks, which are used for the remote stabilization of unstable systems [57]. In supply networks, state-dependent delays appear due to the transportation of materials [51], [52], [55]. In milling processes, speed-dependent delays arise due to the deformation of the cutting tool [1]. The reaction time of a driver is often modeled as a pure [46] or distributed [53] delay. However, the delay depends on the intensity of the disturbance, the size of the tracking error to which the driver is reacting, the speed of the vehicle, the physical situation of the driver, etc. [10]. In irrigation channels, the dynamics of a reach are accurately represented by a time-varying delayed-integrator model [27]. In population dynamics, the time required for the maturation level of a cell to achieve a certain threshold can be modeled as a state-dependent delay [29]. In engine cooling systems, the delay in the distribution of the coolant among the consumers depends on the coolant flow [9]. Finally, models of constant delays can be used to approximate state-dependent delays in chemical process control [22], [39].

B. Literature

Compensation of constant input delays in unstable linear plants is achieved using predictor-based (finite spectrum assignment) techniques [4], [23], [30], [36], [38], [40], [50], [59], [60]. Recent extensions of these designs to linear systems with simultaneous input and state delay are found in [6], [13], and [15]. Predictor-based techniques are developed for linear systems with unknown plant parameters [41], [58] and with unknown delays [5], [7], [8]. Control schemes for nonlinear systems with input delay are developed in [20], [24], [26], [32]–[35], whereas nonlinear systems with state delays are considered in [12], [14], [17]–[19], [31], [47], [48], and [56].

Although there are numerous results for plants with constant input delays, the problem of *compensation* of long time-varying input delays, even for linear systems, is tackled in only a few [4], [25], [43], [44]. Even more rare are papers that deal with the compensation of time-varying input delay in nonlinear systems [17]. No results exist for the compensation of a state-dependent input delay, even for linear plants.

C. Results

We present a methodology for compensating state-dependent input delays for both linear and nonlinear systems. For nonlinear systems with state-dependent input delay which are, in the absence of the input delay, either forward complete and globally stabilizable or just locally stablilizable (by a possible time-varying control law), we design a predictor-based compensator (Section II). Our controller uses predictions of future values of the state on appropriate prediction horizons that depend on the current values of the state. Due to a fundamental restriction on the allowable magnitude of the delay function's gradient (the control signal never reaches the plant if the delay rate is larger than one), we obtain only a regional stability result,

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even for the case of forward-complete systems. We give an estimate of the region of attraction for our control scheme based on the construction of a strict, time-varying Lyapunov functional (Section III). We present a global result for forward-complete systems under a restrictive Lyapunov-like condition, which has to be *a priori* verified, that the delay rate be bounded by unity irrespective of the values of the state and input (Section V). We also deal with linear systems, treating them as a special case of the design for nonlinear systems, for which we prove exponential stability. We present several examples, including a linear example in Section II, illustrating the impossibility of a global result in general, and stabilization of the nonholonomic unicycle subject to distance-dependent input delay in Section IV.

Notation

We use the common definition of class $\mathcal{K}$, $\mathcal{K}_\infty$, and $\mathcal{KL}$ functions from [21]. For an n-vector, the norm $|\cdot|$ denotes the usual Euclidean norm. We say that a function $\rho : \mathbb{R}_+ \times (0,1) \mapsto \mathbb{R}_+$ belongs to class $\mathcal{KC}$ if it is of class $\mathcal{K}$ with respect to its first argument for each value of its second argument and continuous with respect to its second argument. It belongs to class $\mathcal{KC}_\infty$ if it is in $\mathcal{KC}$ and in $\mathcal{K}_\infty$ with respect to its first argument.

II. Problem Formulation and Controller Design

We consider the following system:

$$\dot{X}(t) = f\left(X(t), U\left(t - D\left(X(t)\right)\right)\right) \tag{1}$$

where $X \in \mathbb{R}^n$, $U : [t_0 - D\left(X(t_0)\right), \infty) \to \mathbb{R}$, $t \geq t_0 \geq 0$, $D \in C^1\left(\mathbb{R}^n; \mathbb{R}_+\right)$, $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is locally Lipschitz with $f(0,0) = 0$ and the following holds:

$$\left|f\left(X, \omega\right)\right| \leq \alpha_1\left(|X| + |\omega|\right), \tag{2}$$

for a class $\mathcal{K}_\infty$ function α_1.

Let $P(t)$ denote the value of the state X at the time when the control U applied at time t reaches the plant. We refer to P as the "predictor" state. For systems with constant delays, $D = \text{const}$, it simply holds that $P(t) = X(t+D)$. For systems with state-dependent delays, it is trickier. The time when U reaches the system depends on the value of the state at that time, namely, the following implicit relationship holds $P(t) = X(t + D(P(t)))$ (and $X(t) = P(t - D(X(t)))$). Nevertheless, even for state-dependent delays, $P(t)$ is simply the "predictor" of the future state, at the time when the current control will have an effect on the state. Since $P(t)$ is related to $X(t)$ through an implicit relationship, it is evident that a predictor-based compensation of a state-dependent input delay faces a challenge, which is absent in the case where the delay is merely time-varying. We resolve this challenge by performing our design and analysis using transformations of the time variable $t \mapsto t + D(P(t))$ and $t \mapsto t - D(X(t))$. The difficulty with these transformations, besides not being available explicitly, is that the "prediction horizon" $D(P(t))$ is, in general, different from the delay $D(X(t))$.

We design a predictor-based controller for the plant (1) as

$$U(t) = \kappa\left(\sigma(t), P(t)\right) \tag{3}$$

where for all $t - D\left(X(t)\right) \leq \theta \leq t$

$$P(\theta) = X(t) + \int_{t-D(X(t))}^{\theta} \frac{f\left(P(s), U(s)\right) ds}{1 - \nabla D\left(P(s)\right) f\left(P(s), U(s)\right)} \tag{4}$$

$$\sigma(\theta) = \theta + D\left(P(\theta)\right) \tag{5}$$

for $t \geq t_0$. The initial predictor $P(\theta), \theta \in [t_0 - D\left(X(t_0)\right), t_0]$ is given by (4) for $t = t_0$.

It is somewhat of a challenge to comprehend the mathematical meaning of the relationship (4), where $P(\theta)$ appears on both sides of the equation. It is helpful to start from the linear case, $\dot{X}(t) = AX(t) + BU(t - D)$ with a constant delay D. In that case, the predictor is given explicitly using the variation of constants formula, with the initial condition $P(t - D) = X(t)$, as $P(t) = \exp(AD)X(t) + \int_{t-D}^{t} \exp(A(t-\theta))BU(\theta)d\theta$. For systems that are nonlinear and even for linear systems with a state-dependent delay, $P(t)$ cannot be written explicitly, for the same reason as a nonlinear ODE cannot be solved explicitly. So we represent $P(t)$ implicitly using the nonlinear integral (4). (An additional mathematical explanation of the role that P plays in the analysis, and how it is related to the vector state X and the functional input state $U(\cdot)$, is forthcoming in Fig. 3.)

The computation of $P(t)$ from (4) is straightforward with a discretized implementation where $P(\theta)$ is assigned values based on the right-hand side of (4), which involves earlier values of P and the values of the input U. At each time step, the integral in (4) can be computed using a method of numerical integration (e.g., the trapezoidal rule) with a total number of discrete points N, given by $N(t) = \lfloor D(X(t))/h \rfloor$, where h is the time-descretization step and $\lfloor a \rfloor$ denotes the integer part of a. The implementation of the predictor in (4) might be unsafe when the discretization method of the predictor in (4) is not appropriately chosen [38], whereas a careful discretization yields a safe implementation [37].

To see that $P(t)$ given in (4) is the $\sigma(t) - t = D\left(P(t)\right)$ time units ahead predictor of $X(t)$, differentiate (4) with respect to θ, set $\theta = t$ and perform a change of variables $\tau = \sigma(t)$ in the ODE for $X(\tau)$ given in (1) (where t is replaced by τ), to observe that $P(t)$ satisfies the same ODE in t as $X(\sigma(t))$. Since from (4) for $t = t_0$ and $\theta = t_0 - D\left(X(t_0)\right)$, it follows that $P\left(t_0 - D\left(X(t_0)\right)\right) = X(t_0)$, by defining

$$\phi(t) = t - D\left(X(t)\right), \quad t \geq t_0, \tag{6}$$

$$\sigma(\theta) = \phi^{-1}(\theta), \quad t - D\left(X(t)\right) \leq \theta \leq t \tag{7}$$

we have $P(t_0) = X(\sigma(t_0))$. Hence, indeed $P(t) = X\left(\sigma(t)\right)$ for all $t \geq t_0$.

Noting from (6) and (7) that $D\left(X(\sigma(t))\right) = \sigma(t) - t$, differentiates this equation, we get for all $t - D\left(X(t)\right) \leq \theta \leq t$

$$\frac{d\sigma(\theta)}{d\theta} = \frac{1}{1 - \nabla D\left(P(\theta)\right) f\left(P(\theta), U(\theta)\right)}. \tag{8}$$

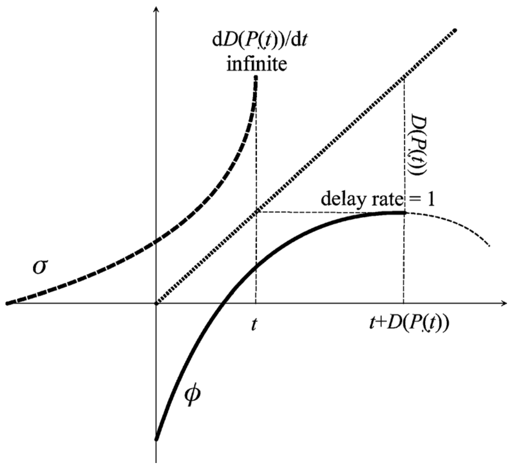

Fig. 1. Functions $\sigma(t)$ (dashed line) and $\phi(t)$ (solid line) in the undesirable case where the infinite slope of $D(P(t))$ causes their noninvertibility.

The possible division by zero in (8) indicates that $\phi(t)$ is not *a priori* guaranteed to be invertible. Another way to see this challenge is to note that the prediction horizon is governed by

$$\frac{dD(P(t))}{dt} = \frac{\nabla D(P(t)) f(P(t), U(t))}{1 - \nabla D(P(t)) f(P(t), U(t))}$$

which, in case of division by zero, can result in the reversal in the direction of the control signal reaching the plant at time $t + D(P(t))$, as displayed in Fig. 1.

Motivated by the need to keep the denominator in (4) and (8) positive, throughout this paper, we consider the condition on the solutions which is given by

$$\mathcal{F}_c : \nabla D(P(\theta)) f(P(\theta), U(\theta)) < c, \quad \text{for all } \theta \geq t_0 - D(X(t_0)) \quad (9)$$

for $c \in (0, 1]$. We refer to $\mathcal{F}_1$ as the *feasibility condition* of the controller (3)–(5).

1) Example 1: We consider a scalar unstable system with a Lyapunov-like delay

$$\dot{X}(t) = X(t) + U\left(t - X(t)^2\right). \quad (10)$$

The delay-compensating controller is

$$U(t) = -2P(t) \quad (11)$$

where for all $\theta \geq -X(0)^2$

$$P(\theta) = X(t) + \int_{t-X(t)^2}^{\theta} \frac{(P(s) + U(s))\, ds}{1 - 2P(s)(P(s) + U(s))}. \quad (12)$$

In Fig. 2, we show the response of the system and the function $\phi(t) = t - X(t)^2$ for four different initial conditions of the state and with the initial conditions for the input chosen as $U(\theta) = 0, -X(0)^2 \leq \theta \leq 0$. We choose $X(0) = 0.15, 0.25, 0.35, X^*$. With X^*, we denote the critical value of $X(0)$ for the given initial condition of the input, such that for any $X(0) \geq X^*$, the control inputs produced by the feedback law (11), (12) for positive t never reach the plant. We calculate this time as follows: The function $\phi(t) = t - X(0)^2 e^{2t}$ has a maximum at t^* if $\log\left(1/\sqrt{2X(0)^2}\right) = t^* > 0$. Since $\phi(t^*) = \log\left(1/\sqrt{2X(0)^2}\right) - 1/2$ has to be positive for the control to reach the plant, it follows $X^* = 1/\sqrt{2e} = 0.43$.

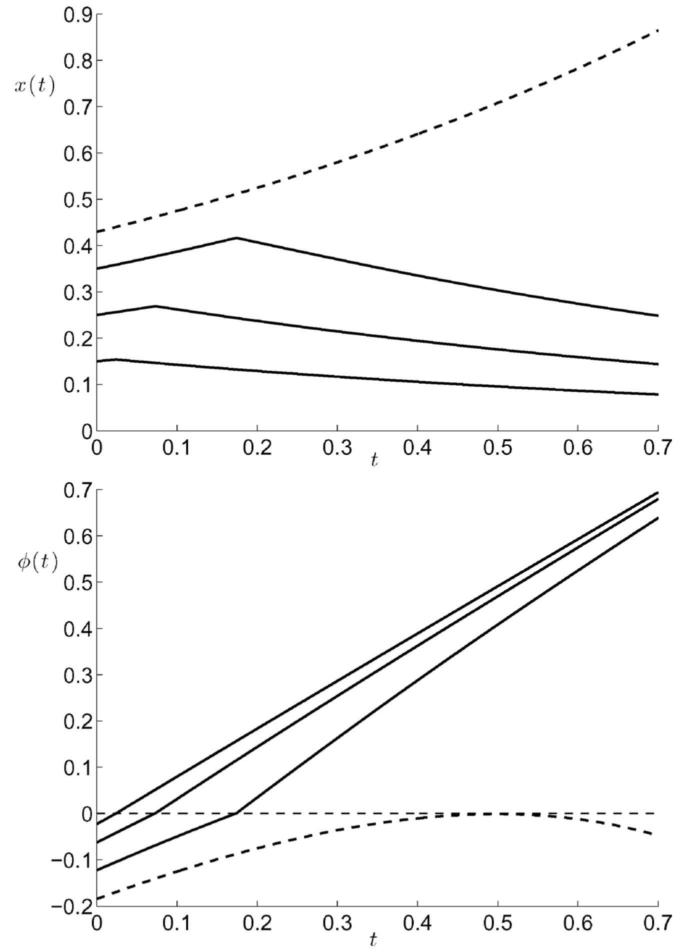

Fig. 2. Response of system (10) with the controller (11)–(12) with initial conditions $U(\theta) = 0$, $-X(0)^2 \leq \theta \leq 0$ and four different initial conditions for the state $X(0) = 0.15, 0.25, 0.35, 0.43$.

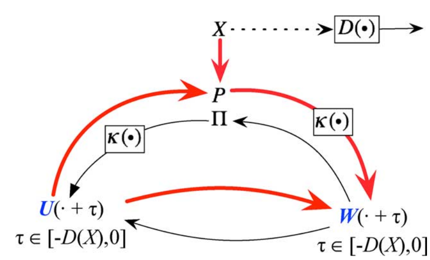

Fig. 3. Interconnections between the predictor states P and Π with the transformations W and U in (21) and (24). The direct backstepping transformation is defined as $(X(t), U(\theta)) \mapsto (X(t), W(\theta))$ and is given in (21), where $P(\theta)$ is given as a function of $X(t)$ and $U(\theta)$ through relations (4)–(5). Analogously, the inverse transformation is defined as $(X(t), W(\theta)) \mapsto (X(t), U(\theta))$ and is given in (24), where $\Pi(\theta)$ is given as a function of $X(t)$ and $W(\theta)$ through relation (25).

III. Stability Analysis for Forward-Complete Nonlinear Systems

Assumption 1: $D \in C^1(\mathbb{R}^n; \mathbb{R}_+)$ and ∇D is locally Lipschitz.[1]

Assumption 2: The plant $\dot{X} = f(X, \omega)$ is strongly forward complete, that is, there exists a smooth positive definite function R and class $\mathcal{K}_\infty$ functions α_2, α_3 and α_4 such that

$$\alpha_2(|X|) \leq R(X) \leq \alpha_3(|X|) \tag{13}$$

$$\frac{\partial R(X)}{\partial X} f(X, \omega) \leq R(X) + \alpha_4(|\omega|) \tag{14}$$

for all $X \in \mathbb{R}^n$ and for all $\omega \in \mathbb{R}$.

This property differs from the standard forward completeness [2] in that we assume that $f(0,0) = 0$ and, hence, $R(\cdot)$ is positive definite. Assumption 2 guarantees that system (1) does not exhibit finite escape time, that is, for every initial condition and every locally bounded input signal, the corresponding solution is defined for all $t \geq t_0$.

Assumption 3: The plant

$$\dot{X}(t) = f(X(t), \kappa(t, X(t)) + \omega(t))$$

satisfies the uniform in the time input-to-state stability property with respect to ω, and the function κ is locally Lipschitz in both arguments and uniformly bounded with respect to its first argument, that is, there exists a class $\mathcal{K}_\infty$ function $\hat{\alpha}$ such that

$$|\kappa(t, \xi)| \leq \hat{\alpha}(|\xi|), \quad \text{for all } t \geq t_0. \tag{15}$$

Theorem 1: Consider the plant (1) together with the control law (3)–(5). Under Assumptions 1, 2, and 3, there exists a class $\mathcal{K}$ function ψ_{RoA}, a class $\mathcal{KC}_\infty$ functions ρ, and a class $\mathcal{KL}$ function β such that for all initial conditions for which U is locally Lipschitz on the interval $[t_0 - D(X(t_0)), t_0)$ and which satisfy

$$B_0(c) : \Omega(t_0) < \psi_{\mathrm{RoA}}(c) \tag{16}$$

for some $0 < c < 1$, where

$$\Omega(t) = |X(t)| + \sup_{t - D(X(t)) \leq \theta \leq t} |U(\theta)| \tag{17}$$

there exists a unique solution to the closed-loop system with X Lipschitz on $[t_0, \infty)$, U Lipschitz on (t_0, ∞), and

$$\Omega(t) \leq \beta(\rho(\Omega(t_0), c), t - t_0) \tag{18}$$

for all $t \geq t_0$. Furthermore, there exists a class $\mathcal{K}$ function δ^* such that for all $t \geq t_0$

$$D(X(t)) \leq D(0) + \delta^*(c) \tag{19}$$

$$\left|\dot{D}(X(t))\right| \leq c. \tag{20}$$

We prove Theorem 1 using Lemmas 1 – 8, which are presented next.

[1]To ensure uniqueness of solutions.

Lemma 1 (Backstepping Transform of Actuator State): The infinite-dimensional backstepping transformation of the actuator state defined for all $t - D(X(t)) \leq \theta \leq t$ by

$$W(\theta) = U(\theta) - \kappa(\sigma(\theta), P(\theta)) \tag{21}$$

together with the predictor-based control law given in relations (3)–(5) transform the system (1) to the "target system" given by

$$\dot{X}(t) = f(X(t), \kappa(t, X(t)) + W(t - D(X(t)))) \tag{22}$$

$$W(t) = 0, \quad \forall t \geq t_0. \tag{23}$$

Proof: Using (3) and the facts that $P(t - D(X(t))) = X(t)$ and $\sigma(t - D(X(t))) = t$, which are immediate consequences of (4)–(5), we get the statement of the lemma. ■

The role of $P(t)$ in our analysis is as an output of a mapping of the state $X(t)$ and input history $U(\theta), \theta \in [t - D, t]$. For instance, in the linear case with a constant delay, this mapping is given explicitly as $P(t) = \exp(AD)X(t) + \int_{t-D}^{t} \exp(A(t - \theta))BU(\theta)d\theta$. In the nonlinear case, the mapping $(X, U) \mapsto P$ is given implicitly by (4). The mapping $(X, U) \mapsto P$ is an intermediate step in transforming the original system (X, U) into the target system (X, W), as displayed in Fig. 3. The transformation $(X, U) \mapsto (X, W)$ is important because the stability analysis can be conducted in the variables (X, W), but not in the original variables (X, U).

Lemma 2 (Inverse Backstepping Transform): The inverse of the infinite-dimensional backstepping transformation defined in (21) is given for all $t - D(X(t)) \leq \theta \leq t$ by

$$U(\theta) = W(\theta) + \kappa(\sigma(\theta), \Pi(\theta)), \tag{24}$$

where for all $t - D(X(t)) \leq \theta \leq t$

$$\Pi(\theta) = \int_{t - D(X(t))}^{\theta} \frac{f(\Pi(s), \kappa(\sigma(s), \Pi(s)) + W(s))}{1 - G(s)} ds + X(t) \tag{25}$$

$$G(s) = \nabla D(\Pi(s)) f(\Pi(s), \kappa(\sigma(s), \Pi(s)) + W(s)). \tag{26}$$

Proof: By direct verification, noting also that $\Pi(\theta) = P(\theta)$ for all $t - D(X(t)) \leq \theta \leq t$, where $\Pi(\theta)$ is driven by the transformed input $W(\theta)$, whereas $P(\theta)$ is driven by the input $U(\theta)$. (See Fig. 3.) ■

It may be slightly puzzling why we present two versions of the predictor P and Π which are, in fact, the same. The reason why we use two distinct symbols for the same quantity is that, in one case, P is expressed in terms of X and U, for the direct backstepping transformation, while, in the other case, Π is expressed in terms of X and W for the inverse backstepping transformation. Since the actual system operates in the (X, U) variables and the analysis is conducted in the (X, W) variables, the direct and backstepping transformations are important.

Lemma 3 (Stability Estimate for Target System): For any positive constant g, there exist a class $\mathcal{K}_\infty$ function δ_1 and a class $\mathcal{KL}$ function β_2 such that for all solutions of the system satisfying (9) for $0 < c < 1$, the following holds:

$$\Xi(t) \leq \beta_2(\rho_*(\Xi(t_0), c), t - t_0) \tag{27}$$

$$\Xi(t) = |X(t)| + \sup_{t - D(X(t)) \leq \theta \leq t} |W(\theta)| \tag{28}$$

for all $t \geq t_0$, where

$$\rho_*(s,c) = \frac{e^{g/(1-c)}}{1-c} e^{(g/(1-c))(D(0)+\delta_1(s))} s. \tag{29}$$

Proof: Based on Assumption 3 and [16] (Remark 3.2), there exists a smooth function $S(t, X(t)) : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}_+$ and class $\mathcal{K}_\infty$ functions α_5, α_6, α_7, and α_8 such that

$$\alpha_5(|X(t)|) \leq S(t, X(t)) \leq \alpha_6(|X(t)|) \tag{30}$$

$$\dot{S}(t, X(t)) \leq -\alpha_7(|X(t)|) + \alpha_8(|W(t-D(X(t)))|) \tag{31}$$

$$\dot{S}(t, X(t)) = \frac{\partial S(t, X(t))}{\partial t} + \frac{\partial S(t, X(t))}{\partial X} \times f(X(t), \kappa(t, X(t)) + W(t - D(X(t)))). \tag{32}$$

Consider now the following Lyapunov functional for the "target system" given in (22) and (23)

$$V(t) = S(t, X(t)) + k \int_0^{L(t)} \frac{\alpha_8(r)}{r} dr \tag{33}$$

where

$$\begin{aligned} L(t) &= \sup_{t-D(X(t)) \leq \theta \leq t} \left| e^{g(1+\sigma(\theta)-t)} W(\theta) \right| \\ &= \lim_{n\to\infty} \left(\int_{t-D(X(t))}^{t} e^{2ng(1+\sigma(\theta)-t)} W(\theta)^{2n} d\theta \right)^{1/2n} \end{aligned} \tag{34}$$

with $g > 0$. We now upperbound and lowerbound $L(t)$ in terms of $\sup_{t-D(X(t))\leq\theta\leq t} |W(\theta)|$. From (9) for $0 < c < 1$ we get that $\dot{\sigma}(\theta) \leq 1/(1-c)$. Integrating the relation $\dot{\sigma}(\theta) \leq 1/(1-c)$ from $t - D(X(t))$ to θ and since $\sigma(t - D(X(t))) = t$, we have for all $t - D(X(t)) \leq \theta \leq t$

$$1 + \sigma(\theta) - t \leq \frac{1 - c + D(X(t))}{1-c}. \tag{35}$$

Since $D \in C^1(\mathbb{R}^n; \mathbb{R}_+)$ there exists a function $\delta_1 \in \mathcal{K}_\infty \cap C^1$ such that

$$D(X) \leq D(0) + \delta_1(|X|). \tag{36}$$

Therefore

$$L(t) \leq \frac{e^{(g/(1-c))(1+D(0)+\delta_1(|X(t)|))}}{1-c} \sup_{t-D(X(t))\leq\theta\leq t} |W(\theta)|. \tag{37}$$

Similarly, using the fact that $\sigma(t - D(X(t))) - t = 0$ and since $\sigma(\theta)$ is increasing we obtain

$$1 \leq 1 + \sigma(\theta) - t, \quad t - D(X(t)) \leq \theta \leq t. \tag{38}$$

Therefore, with the help of (38), we have

$$L(t) \geq e^g \sup_{t-D(X(t))\leq\theta\leq t} |W(\theta)|. \tag{39}$$

Taking the time derivative of $L(t)$, with (23) we obtain

$$\begin{aligned} \dot{L}(t) = \lim_{n\to\infty} \frac{1}{2n} &\left(\int_{t-D(X(t))}^{t} e^{2ng(1+\sigma(\theta)-t)} W(\theta)^{2n} d\theta \right)^{1/2n-1} \\ &\times \left(-\left(1 - \frac{dD(X(t))}{dt}\right) e^{2ng} W(t - D(X(t)))^{2n} \right. \\ &\left. - 2ng \int_{t-D(X(t))}^{t} e^{2ng(1+\sigma(\theta)-t)} W(\theta)^{2n} d\theta \right). \end{aligned} \tag{40}$$

Using (9), we have $dD(X(t))/dt < 1$ and, hence, $\dot{L}(t) \leq -gL(t)$. With this inequality and (31), taking the derivative of (33) we obtain $\dot{V}(t) \leq -\alpha_7(|X(t)|) + \alpha_8(|W(t-D(X(t)))|) - kg\alpha_8(L(t))$. With the help of (39) and choosing $k = 2/g$, we get $\dot{V}(t) \leq -\alpha_7(|X(t)|) - \alpha_8(L(t))$. Using (30), the definition of $L(t)$ in (34) and (33), we conclude that there exists a class $\mathcal{K}$ function γ_1 such that $\dot{V}(t) \leq -\gamma_1(V(t))$. Using the comparison principle and [21, Lemma 4.4], there exists a class $\mathcal{KL}$ function β_1 such that $V(t) \leq \beta_1(V(t_0), t - t_0)$. Using (30), the definition of $V(t)$ in (33) and the properties of class $\mathcal{K}$ functions, we arrive at $|X(t)| + L(t) \leq \beta_2(|X(t_0)| + L(t_0), t - t_0)$ for some class $\mathcal{KL}$ function β_2. Using relations (37) and (39), the lemma is proved. ■

There is conservativeness involved in passing between (X, U) and (X, W) in both directions, just like there is a loss to a currency exchange customer both when buying and selling. With Lemma 4, we quantify a bound when going from (X, U) to (X, W) via P, and with Lemma 5, we quantify a bound when going back from (X, W) to (X, U) via Π.

Lemma 4 (Bound on the Predictor in Terms of Actuator State): There exists a class $\mathcal{KC}_\infty$ function ρ_1 such that for all solutions of the system satisfying (9) for $0 < c < 1$, the following holds for all $t - D(X(t)) \leq \theta \leq t$:

$$|P(\theta)| \leq \rho_1\left(|X(t)| + \sup_{t-D(X(t))\leq s\leq t} |U(s)|, c\right). \tag{41}$$

Proof: Consider the following ODE in θ which follows by differentiating (4):

$$\frac{dP(\theta)}{d\theta} = \frac{f(P(\theta), U(\theta))}{1 - \nabla D(P(\theta)) f(P(\theta), U(\theta))}, \quad t - D(X(t)) \leq \theta \leq t. \tag{42}$$

With the change of variables

$$y = \sigma(\theta) \tag{43}$$

we rewrite (42) as

$$\frac{dP(\phi(y))}{dy} = f(P(\phi(y)), U(y - D(P(\phi(y))))), \quad t \leq y \leq \sigma(t). \tag{44}$$

Using (14), we get

$$\frac{dR(P(\phi(y)))}{d\theta} \frac{d\theta}{dy} \leq R(P(\phi(y))) + \alpha_4(|U(y - D(P(\phi(y))))|). \tag{45}$$

With (9), we have for all $t - D(X(t)) \leq \theta \leq t$

$$\frac{dR(P(\theta))}{d\theta} \leq \frac{1}{1-c}\left(R(P(\theta)) + \alpha_4(|U(\theta)|)\right). \tag{46}$$

Using the comparison principle and (36), one gets

$$R(P(\theta)) \leq e^{(D(0)+\delta_1(|X(t)|))/(1-c)}\Bigg(R(X(t)) + \sup_{t-D(X(t))\leq s\leq t} \alpha_4(|U(s)|)\Bigg) \tag{47}$$

for $t - D(X(t)) \leq \theta \leq t$. With standard properties of class $\mathcal{K}_\infty$ functions, we get the statement of the lemma where the class $\mathcal{KC}_\infty$ function ρ_1 is given as

$$\rho_1(s,c) = \alpha_2^{-1}\left((\alpha_3(s)+\alpha_4(s))\, e^{(D(0)+\delta_1(s))/(1-c)}\right). \tag{48}$$

■

Lemma 5: (Bound on Predictor in Terms of Transformed Actuator State): There exists a class $\mathcal{K}$ function γ_4 such that for all solutions of the system satisfying (9) for $0 < c < 1$, the following holds for all $t - D(X(t)) \leq \theta \leq t$:

$$|\Pi(\theta)| \leq \gamma_4\left(|X(t)| + \sup_{t-D(X(t))\leq s\leq t} |W(s)|\right). \tag{49}$$

Proof: Under Assumption 3 and [16], there exists class $\mathcal{KL}$ function β_3 and class $\mathcal{K}$ function γ_2 such that

$$|Y(\tau)| \leq \beta_3(|Y(t_0)|, \tau - t_0) + \gamma_2\left(\sup_{s\geq t_0}|\omega(s)|\right), \quad \tau \geq t_0 \tag{50}$$

where $Y(\tau)$ is the solution of $\dot{Y}(\tau) = f(Y(\tau), \kappa(\tau, Y(\tau)) + \omega(\tau))$. Using the change of variable (43) and definitions (25), (44), we have

$$\frac{d\Pi(\phi(y))}{dy} = f\left(\Pi(\phi(y)), \kappa(y, \Pi(\phi(y))) + W(\phi(y))\right), \quad t \leq y \leq \sigma(t). \tag{51}$$

Using (50), we have

$$|\Pi(\theta)| \leq \gamma_3(|X(t)|) + \gamma_2\left(\sup_{t-D(X(t))\leq s\leq t}|W(s)|\right), \quad t - D(X(t)) \leq \theta \leq t \tag{52}$$

where the class $\mathcal{K}$ function γ_3 is defined as $\gamma_3(s) = \beta_3(s, 0)$. With the properties of class $\mathcal{K}$ functions, we obtain the statement of the lemma, where $\gamma_4(s) = \gamma_2(s) + \gamma_3(s)$ is of class $\mathcal{K}$. ■

Lemma 6: (Equivalence of Norms for Original and Target System): There exists a function ρ_2 of class $\mathcal{KC}_\infty$ and a class $\mathcal{K}_\infty$ function α_9 such that for all solutions of the system satisfying (9) for $0 < c < 1$, the following holds:

$$\Omega(t) \leq \alpha_9^{-1}(\Xi(t)) \tag{53}$$

$$\Xi(t) \leq \rho_2(\Omega(t), c) \tag{54}$$

for all $t \geq t_0$, where Ω and Ξ are defined in (17) and in (28), respectively.

Proof: With the inverse backstepping transformation (24) and the bound (49), we get the bound (53) with $\alpha_9^{-1}(s) = s + \hat{\alpha}\circ\gamma_4(s)$. Using the direct backstepping transformation (21) and the bound (41), we get the bound (54) with $\rho_2(s,c) = s + \hat{\alpha}(\rho_1(s,c))$. ■

Lemma 7: (Ball Around the Origin Within the Feasibility Region): There exists a function $\bar{\rho}_c$ of class $\mathcal{KC}_\infty$ such that all of the solutions that satisfy

$$\bar{B}(c): |X(t)| + \sup_{t-D(X(t))\leq\theta\leq t}|U(\theta)| < \bar{\rho}_c(c,c), \qquad t \geq t_0 \tag{55}$$

for $0 < c < 1$, also satisfy (9).

Proof: Since $D \in C^1(\mathbb{R}^n; \mathbb{R}_+)$, there exists a class $\mathcal{K}_\infty$ function δ_2 such that[2]

$$|\nabla D(X)| \leq |\nabla D(0)| + \delta_2(|X|). \tag{56}$$

If a solution satisfies

$$(|\nabla D(0)| + \delta_2(|P(\theta)|)) \times \alpha_1\Bigg(|P(\theta)| + \sup_{t-D(X(t))\leq s\leq t}|U(s)|\Bigg) < c, \quad t - D(X(t)) \leq \theta \leq t \tag{57}$$

for $0 < c < 1$, then it also satisfies (9). Using Lemma 4, we conclude that (57) is satisfied for $0 < c < 1$ as long as (55) holds, where the class $\mathcal{KC}_\infty$ function ρ_c is defined as

$$\rho_c(s,c) = (|\nabla D(0)| + \delta_2(\rho_1(s,c)))\, \alpha_1((\rho_1(s,c) + s)) \tag{58}$$

and with $\bar{\rho}_c$, we denote the inverse function of ρ_c with respect to ρ_c's first argument. ■

Lemma 8 (Estimate of the Region of Attraction): There exists a class $\mathcal{K}$ function ψ_{RoA} such that for all initial conditions of the closed-loop system (1), (3), (4), (5) that satisfy relation (16), the solutions of the system satisfy (55) for $0 < c < 1$ and, hence, satisfy (9).

Proof: Using Lemma 6 with the help of (27), we have

$$\Omega(t) \leq \alpha_9^{-1}\left(\beta_2\left(\rho_*\left(\rho_2(\Omega(t_0), c), c\right), t - t_0\right)\right) \tag{59}$$

where Ω is defined in (17). By defining the class $\mathcal{K}_\infty$ function α_{10} as $\alpha_{10}(s) = \alpha_9^{-1}(\beta_2(s,0))$, we obtain

$$\Omega(t) \leq \alpha_{10}\left(\rho_*\left(\rho_2(\Omega(t_0)), c\right), c\right). \tag{60}$$

Hence, for all initial conditions that satisfy the bound (16) with any class $\mathcal{K}$ choice $\psi_{\mathrm{RoA}}(c) \leq \bar{\psi}^*{}_{\mathrm{RoA}}(\bar{\rho}_c(c,c), c)$, where $\bar{\psi}^*{}_{\mathrm{RoA}}(s,c)$ is the inverse of the class $\mathcal{KC}_\infty$ function $\psi^*{}_{\mathrm{RoA}}(s,c) = \alpha_{10}(\rho^*(\rho_2(s,c),c))$ with respect to ψ^*_{RoA}'s

[2]Estimate (56) is derived based on the nonrestrictive assumption that the delay is a continuously differentiable function of X. Using bound (56), one can restrict the gradient of the delay D (which is needed for (9)) by restricting the size of the state X. This enables one to estimate the region of attraction of the proposed control law (see Lemma 8).

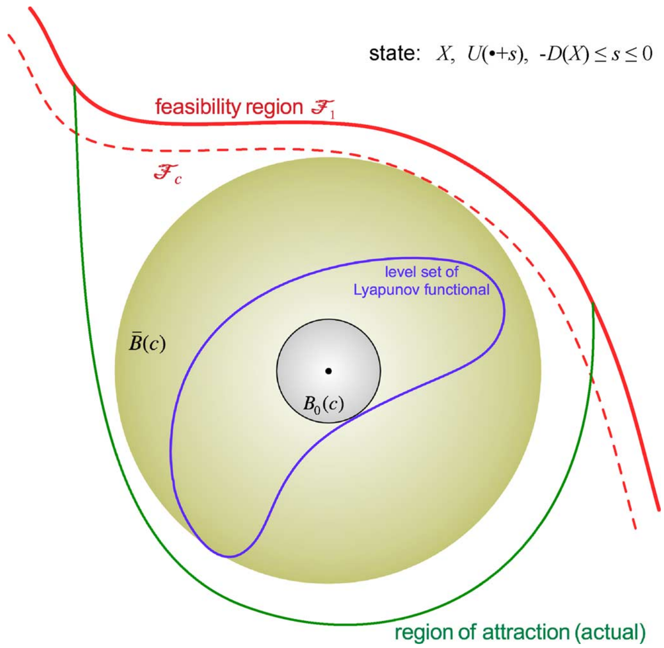

Fig. 4. Sets arising in the proof of Theorem 1 in the infinite-dimensional state space $\mathbb{R}^n \times C[t-D(X(t)),t)$. $B_0(c)$: the ball of initial conditions allowed in the proof of the theorem. $\bar{B}(c)$: the ball inside where the ensuing solutions are trapped.

first argument, the solutions satisfy (55). Furthermore, for all of those initial conditions, the solutions verify (9) for all $\theta \geq t_0 - D(X(t_0))$. Fig. 4 illustrates the set relationships in the proof. ■

Proof of Theorem 1: Using (59), we obtain (18) with $\beta(s,t) = \alpha_9^{-1}(\beta_2(s,t))$ and $\rho(s,c) = \rho_*(\rho_2(s,c),c)$. From (1), the Lipschitzness of U on $[t_0 - D(X(t_0)), t_0)$ guarantees the existence and uniqueness of $X \in C^1[t_0, \sigma^*)$, where $\sigma^* = t_0 + D(X(\sigma^*))$, the system (22), (23) guarantees the existence and uniqueness of $X \in C^1(\sigma^*, \infty)$, and the boundedness of W and (22) guarantee that X is continuous at $t = \sigma^*$. By integrating (22) between any two time instants, it is shown that X is Lipschitz on $[t_0, \infty)$ with a Lipschitz constant given by a uniform bound on the right-hand side of (22). Since $U(t) = \kappa(\sigma(t), \Pi(t))$, where $\sigma(t) = t + D(\Pi(t))$ and

$$\dot{\Pi}(t) = \frac{f(\Pi(t), \kappa(\sigma(t), \Pi(t)))}{1 - \nabla D(\Pi(t)) f(\Pi(t), \kappa(\sigma(t), \Pi(t)))}$$

for $t \geq t_0$, Assumption 1 (Lipschitzness of ∇D), Assumption 3 (Lipschitzness of κ in both arguments), and (9) ensure that the right-hand side of the Π-ODE is Lipschitz, which guarantees that $\Pi \in C^1(t_0, \infty)$. Since $\kappa(t + D(\Pi(t)), \Pi(t))$ is Lipschitz in t on (t_0, ∞), so is U. Using Lemma 8, (36), and (56), we get (19) and (20) with any class $\mathcal{K}$ function $\delta^*(c) \geq \delta_1(\bar{\rho}_c(c,c))$. ■

Remark 1: The proof of the existence, uniqueness, and regularity portion of Theorem 1 is rather straightforward because the input function $U(\cdot)$, which includes the dependence on the state $X(t)$, is a part of the system's vector field. This is in contrast with the existence analysis in [45], which is considerably more complex, despite $X(t)$ being scalar, because the state-dependent delay acts on the state rather than on an input. Paradoxically, even though control design is typically vastly more difficult for systems with delays on the input than on the state, the existence analysis is easier when the delay affects only the input and not the state, even when the delay is state dependent.

We note here that for linear plants, that is, when system (1) is

$$\dot{X}(t) = AX(t) + BU(t - D(X(t))). \tag{61}$$

Assumption 3 is satisfied when the pair (A, B) is stabilizable and Assumption 2 is satisfied for any A by means of

$$\frac{d|Y(\tau)|^2}{d\tau} \leq (2|A|+1)|Y(\tau)|^2 + |B|^2\omega^2(\tau). \tag{62}$$

The controller for the linear case is

$$U(t) = KP(t) \tag{63}$$

with the predictor $P(t)$ given by

$$P(\theta) = \int_{t-D(X(t))}^{\theta} \frac{(AP(s) + BU(s))\, ds}{1 - \nabla D(P(s))(AP(s) + BU(s))} + X(t) \tag{64}$$

for $t - D(X(t)) \leq \theta \leq t$. The predictor $P(\theta)$ is not given explicitly even for the linear case. We next establish the following result with explicit estimates that highlight the nonlinear role of the delay function $D(X)$ and exponential decay in time.

Theorem 2: Consider the plant (61) together with the control law (63), (64), and K chosen such that $A + BK$ is Hurwitz, namely, $(A+BK)^T P + P(A+BK) = -Q$, for some $P = P^T > 0$ and $Q = Q^T > 0$. Under Assumption 1, for all initial conditions of the plant that satisfy

$$|X(t_0)| + \sup_{t_0 - D(X(t_0)) \leq \theta \leq t_0} |U(\theta)| < \zeta_{\mathrm{RoA}}(c) \tag{65}$$

for some $0 < c < 1$, the following holds:

$$\Omega(t) \leq \zeta_4(\Omega(t_0), c)\, e^{-\lambda(t-t_0)} \tag{66}$$

for all $t \geq t_0$, where Ω is defined in (17), $\lambda > 0$, $\zeta_4 \in \mathcal{KC}_\infty$, and $\zeta_{\mathrm{RoA}} \in \mathcal{K}$ are given by

$$\zeta_{\mathrm{RoA}}(c) \leq \bar{\zeta}_4\left(\bar{\zeta}_c(c,c), c\right) \tag{67}$$

$$\zeta_4(s,c) = (1 + |K| r_1)\sqrt{\frac{2}{\min\{\lambda_{\min}(P), k\}}} \times \sqrt{\zeta_2(s + |K|\zeta_1(s,c), c)} \tag{68}$$

$$\zeta_c(c,s) = (|A| + |B|)(|\nabla D(0)| + \delta_2(\zeta_1(c,s))) \times (\zeta_1(c,s) + s) \tag{69}$$

$$\zeta_1(s,c) = \left(1 + \frac{|B|^2}{2|A|+1}\right)^{1/2} e^{((2|A|+1)/(2(1-c)))(D(0)+\delta_1(s))} s \tag{70}$$

$$\zeta_2(s,c) = \left(\lambda_{\max}(P) + \frac{k e^{(2g/(1-c))(1+D(0)+\delta_1(s))}}{(1-c)^2}\right) s^2 \tag{71}$$

$$k = \frac{1}{g}\left(\frac{2|PB|}{\lambda_{\min}(Q)} + \frac{\lambda_{\min}(Q)}{4}\right) \tag{72}$$

$$r_1 = \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\left(1 + 4\frac{\lambda_{\max}(P)|PB|^2}{\lambda_{\min}(Q)^2}\right) \tag{73}$$

$$\lambda = \frac{\lambda_{\min}(Q)}{4\max\{k, \lambda_{\max}(P)\}} \tag{74}$$

and g is an arbitrary positive constant. Furthermore, inequalities (19) and (20) hold with $\delta^*(c) \geq \delta_1\left(\bar{\zeta}_c(c,c)\right)$.

Proof: For the special case of linear systems, Lemma 3 is applied using the Lyapunov function

$$V(t) = X(t)^T P X(t) + kL(t)^2. \quad (75)$$

Using the fact that $\dot{L}(t) \leq -gL(t)$, together with (39) and (37), we get

$$\dot{V}(t) \leq -r_2 V(t) \quad (76)$$

where $r_2 = \lambda_{\min}(Q)/2\max\{k, \lambda_{\max}(P)\}$. With relations (37), (39), (75), (76), and (9) for $0 < c < 1$ we establish that

$$\Xi(t) \leq \sqrt{\frac{2}{r_3}} e^{-(r_2/2)(t-t_0)} \sqrt{\zeta_2\left(\Xi(t_0), c\right)} \quad (77)$$

where Ξ is defined in (28). Using (48) together with (62), relation (41) holds with $\rho_1(s,c)$ replaced by $\zeta_1(s,c)$. Using (49) with $\gamma_4(s) = r_1 s$, Lemma 6 applies with

$$\rho_2(s,c) = s + |K|\zeta_1(s,c) \quad (78)$$

$$\alpha_9^{-1}(s) = \frac{1}{1+|K|r_1} s. \quad (79)$$

As in the proof of Lemma 7, condition $\mathcal{F}_c$ for $0 < c < 1$ in (9) is satisfied when

$$|X(t)| + \sup_{t-D(X(t)) \leq \theta \leq t} |U(\theta)| < \bar{\zeta}_c(c,c) \quad (80)$$

where with $\bar{\zeta}_c$, we denote the inverse function of the class $\mathcal{KC}_\infty$ function $\zeta_c(c,s)$ with respect to its first argument. Using Lemma 6 together with(78) and (79) and Lemma 3 together with (77), we obtain (66). Using (65) and (66) with $t = t_0$, we get (80). Bounds (19)–(20) follow from (80), (36). ■

IV. Example: Nonholonomic Unicycle Subject to Distance-Dependent Input Delay

We consider the problem of stabilizing a mobile robot modeled as

$$\dot{x}(t) = v\left(t - D\left(x(t), y(t)\right)\right)\cos\left(\theta(t)\right) \quad (81)$$

$$\dot{y}(t) = v\left(t - D\left(x(t), y(t)\right)\right)\sin\left(\theta(t)\right) \quad (82)$$

$$\dot{\theta}(t) = \omega\left(t - D\left(x(t), y(t)\right)\right) \quad (83)$$

subject to an input delay that grows with the distance relative to the reference position as

$$D\left(x(t), y(t)\right) = x(t)^2 + y(t)^2 \quad (84)$$

where $(x(t), y(t))$ is the position of the robot, $\theta(t)$ is heading, $v(t)$ is speed, and $\omega(t)$ is the turning rate. When $D = 0$, a time-varying stabilizing controller for this system is proposed in [49] as

$$\omega(t) = -5P(t)^2\cos\left(3\sigma(t)\right) - P(t)Q(t) \times \left(1 + 25\cos\left(3\sigma(t)\right)^2\right) - \Theta(t) \quad (85)$$

$$v(t) = -P(t) + 5Q(t)\left(\sin\left(3\sigma(t)\right) - \cos\left(3\sigma(t)\right)\right) + Q(t)\omega(t) \quad (86)$$

$$P(t) = X(t)\cos\left(\Theta(t)\right) + Y(t)\sin\left(\Theta(t)\right) \quad (87)$$

$$Q(t) = X(t)\sin\left(\Theta(t)\right) - Y(t)\cos\left(\Theta(t)\right) \quad (88)$$

with

$$X = x, \quad Y = y, \quad \Theta = \theta, \quad \sigma(t) = t. \quad (89)$$

The proposed method replaces (89) with

$$X(t) = x(t) + \int_{t-D(x(t),y(t))}^{t} \dot{\sigma}(s) v(s)\cos\left(\Theta(s)\right) ds \quad (90)$$

$$Y(t) = y(t) + \int_{t-D(x(t),y(t))}^{t} \dot{\sigma}(s) v(s)\sin\left(\Theta(s)\right) ds \quad (91)$$

$$\Theta(t) = \theta(t) + \int_{t-D(x(t),y(t))}^{t} \dot{\sigma}(s)\omega(s) ds \quad (92)$$

$$\sigma(t) = t + D\left(X(t), Y(t)\right) \quad (93)$$

$$\dot{\sigma}(s) = \frac{1}{1 - 2v(s)\left(X(s)\cos(\Theta(s)) + Y(s)\sin(\Theta(s))\right)}. \quad (94)$$

The initial conditions are chosen as $x(0) = y(0) = \theta(0) = 1$ and $\omega(s) = v(s) = 0$ for all $-x(0)^2 - y(0)^2 \leq s \leq 0$. From the given initial conditions, we get the initial conditions for the predictors (90)–(92) as $X(s) = Y(s) = \Theta(s) = 1$ for all $-2 \leq s \leq 0$. From the above initial conditions for the predictors, one can verify that the system initially lies inside the feasibility region. The controller "kicks in" at the time instant t_0 at which $t_0 = x(t_0)^2 + y(t_0)^2$. Since $v(s) = \omega(s) = 0$ for $s < 0$, we conclude that $x(t) = y(t) = \theta(t) = 1$ for all $0 \leq t \leq t_0$ and, hence, $t_0 = 2$. In Fig. 5, we show the trajectory of the robot in the xy plane, whereas in Fig. 6, we show the resulting state-dependent delay and the controls $v(t)$ and $\omega(t)$. In the case of the uncompensated controller (85)–(88), (89), the system is unstable, the delay grows approximately linearly in time, and the vehicle's trajectory is a divergent Archimedean spiral. The compensated controller (85)–(88), (90)–(94) recovers the delay-free behavior after 2 s. From Fig. 5, one can also conclude that the heading $\theta(t)$ in the case of the compensated controller converges to zero with damped oscillations, whereas in the case of the uncompensated controller, it increases toward negative infinity (the robot moves clockwise on a spiral).

V. Global Stabilization

The key challenge for the stabilization of systems with state-dependent input delay is to maintain the feasibility condition (9) (i.e., to keep the delay rate below one). This condition can be satisfied *a priori* by making the following restrictive but verifiable assumption.

Assumption 4: $\nabla D(X) f(X,\omega) < c$ for some $0 < c < 1$ and all $(X,\omega) \in \mathbb{R}^{n+1}$.

Corollary 1: Consider the plant (1) together with the control law (3)–(5). Under Assumptions 1, 2, 3, and 4 there exists a class $\mathcal{KL}$ function β_{g} and a class $\mathcal{KC}_\infty$ function ρ_{g} such that

$$\Omega(t) \leq \beta_{\mathrm{g}}\left(\rho_{\mathrm{g}}\left(\Omega(t_0), c\right), t - t_0\right) \quad (95)$$

for all $t \geq t_0$ and some $0 < c < 1$, where Ω is defined in (17).

1) Example 2: We consider the scalar system

$$\dot{X}(t) = \frac{X(t) + U\left(t - D\left(X(t)\right)\right)}{U\left(t - D\left(X(t)\right)\right)^2 + 1} = \quad (96)$$

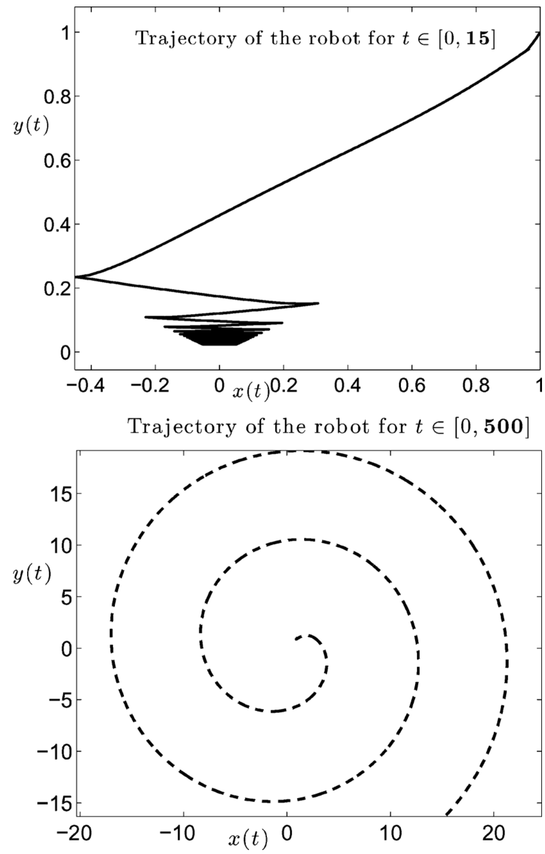

Fig. 5. Trajectory of the robot, with the compensated controller (85)–(88), (90)–(94) (solid line), and the uncompensated controller (85)–(88), (89) (dashed line) with initial conditions $x(0) = y(0) = \theta(0) = 1$ and $\omega(s) = v(s) = 0$ for all $-x(0)^2 - y(0)^2 \leq s \leq 0$.

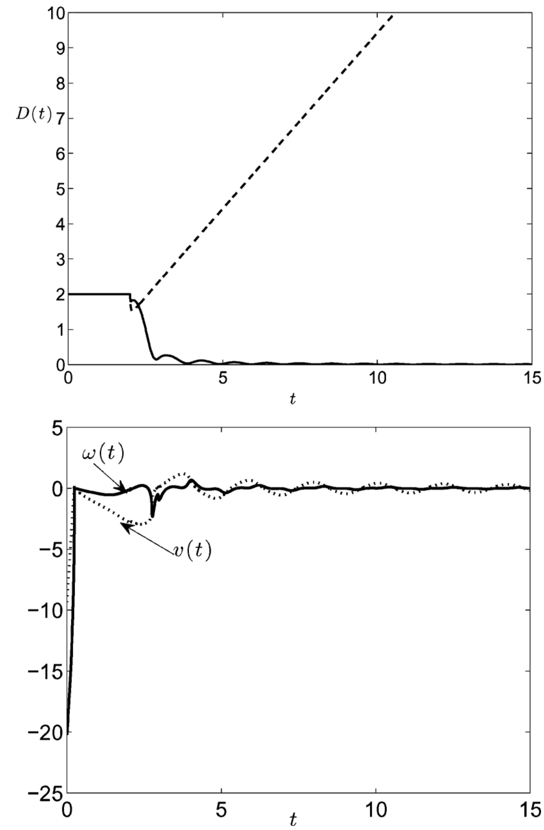

Fig. 6. Left: The delay with the controller (85)–(88), (90)–(94) (solid line) and the controller (85)–(88), (89) (dashed line) with initial conditions $x(0) = y(0) = \theta(0) = 1$ and $\omega(s) = v(s) = 0$ for all $-x(0)^2 - y(0)^2 \leq s \leq 0$. Right: The control efforts $v(t)$ and $\omega(t)$ with the controller (85)–(88), (90)–(94) with initial conditions $x(0) = y(0) = \theta(0) = 1$ and $\omega(s) = v(s) = 0$ for all $-x(0)^2 - y(0)^2 \leq s \leq 0$.

with

$$D(X(t)) = \frac{1}{4}\log\left(X(t)^2+1\right). \tag{97}$$

Taking the time derivative of $D(X(t))$ and using Young's inequality, one gets that

$$\frac{dD(X(t))}{dt} \leq \frac{\frac{3}{4}X(t)^2 + \frac{1}{4}U\left(t - D(X(t))\right)^2}{\left(X(t)^2+1\right)\left(U\left(t-D(X(t))\right)^2+1\right)} \leq \frac{6}{7}. \tag{98}$$

Since system (96) and (97) satisfies Assumption 1, 2, 3, and 4, Corollary 1 applies. The control law has the form

$$U(t) = -2P(t) \tag{99}$$

$$P(t) = X(t) + \int_{t-D(X(t))}^{t} \frac{g_1(\theta)d\theta}{g_2(\theta)} \tag{100}$$

$$g_1(\theta) = 2\left(P(\theta)^2+1\right)\left(P(\theta)+U(\theta)\right) \tag{101}$$

$$g_2(\theta) = 2\left(U(\theta)^2+1\right)\left(P(\theta)^2+1\right) - P(\theta)\left(P(\theta)+U(\theta)\right) \tag{102}$$

where D is given in (97). In Fig. 7, we show the response of the system with initial conditions as $X(0) = 1.5$, $U(\theta) = 0$ and

$$P(\theta) = X(0) + \int_{-(1/4)\log(X(0)^2+1)}^{0} \frac{2\left(P(\theta)^2+1\right)P(\theta)d\theta}{\left(P(\theta)^2+2\right)}$$

for all $-(1/4)\log\left(X(0)^2+1\right) \leq \theta \leq 0$. Initially, $X(t)$ runs in open loop and grows exponentially, while $D(t)$ grows roughly linearly, because of (97). This goes on until control "kicks in" at $t^* = (1/4)\log\left(1 + X(0)^2 e^{2t^*}\right) = 0.4835$. For $t > t^*$, the controller starts bringing $X(t)$ back to zero. As $X(t)$ decays according to the target system $\dot{X}(t) = -X(t)/(1+4X(t)^2)$, the delay $D(t)$ also decays. Starting from $X(t^*)$, $X(t)^2$ first decays roughly linearly in t, making the decay of $D(t)$ logarithmic. When $X(t)$ becomes small, its decay becomes exponential and the decay of $D(t)$ is also exponential. Initially, $P(t)^2$ decays linearly in t and later $P(t)$ decays exponentially. The decay of $U(t) = -2P(t)$ follows the same pattern as $P(t)$.

VI. Stability Analysis for Locally Stabilizable Nonlinear Systems

In Section III we proved a local stability result under assumptions of global stabilizability (Assumption 3) and forward completeness (Assumption 2) of the delay-free system. It is reasonable to ask whether a local stability result can be established under a less restrictive assumption of local stabilizability of the delay-free system. In this section we provide an affirmative answer to this question. Our proof of this result does not employ a Lyapunov construction and, as such, provides an illustrative alternative to the proof technique in Section III.

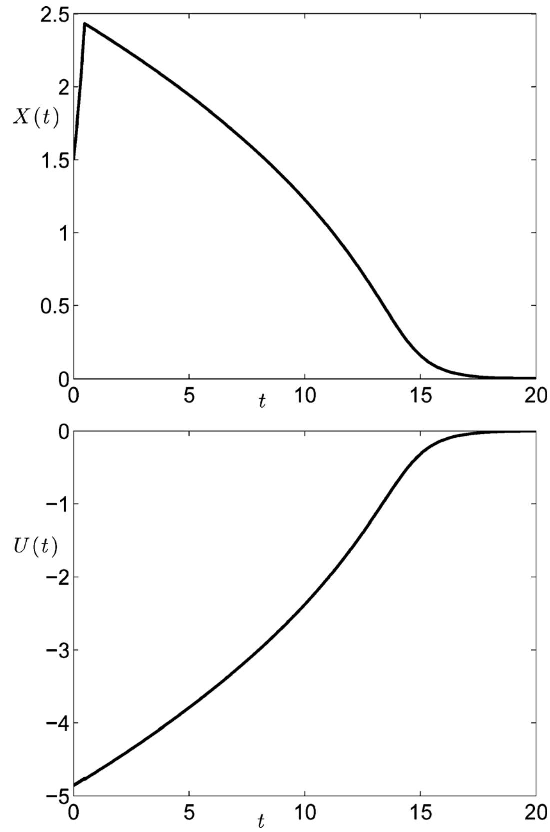

Fig. 7. Response of system (96) with the controller (99)–(100) and initial conditions $X(0)=1.5, U(\theta)=0$, for all $-(1/4)\log\left(X(0)^2+1\right)\le\theta\le 0$.

Assumption 5: There exists a locally Lipschitz feedback controller $\kappa:\mathbb{R}_+\times\mathbb{R}^n\mapsto\mathbb{R}$ that satisfies (15), a positive constant R and a class $\mathcal{KL}$ function β^* such that for the system $\dot{Y}(t)=f(Y(t),\kappa(t,Y(t)))$, the following holds for all $|Y(t_0)|\le R$:

$$|Y(t)|\le\beta^*\left(|Y(t_0)|,t-t_0\right),\quad t\ge t_0. \tag{103}$$

Theorem 3: Consider the plant (1), together with the predictor controller (3)–(5), and define Ω as in (17). Under Assumptions 1 and 5, there exists a function $\psi_{\rm RoA}$ of class $\mathcal{K}$ and a class $\mathcal{KL}$ function $\hat{\beta}$ such that for all initial conditions for which U is locally Lipschitz on the interval $[t_0-D\left(X(t_0)\right),t_0)$ and which satisfy

$$\Omega(t_0)<\psi_{\rm RoA}(c) \tag{104}$$

for some $0<c<1$, there exists a unique solution to the closed-loop system with X Lipschitz on $[t_0,\infty)$, U Lipschitz on (t_0,∞), and

$$\Omega(t)\le\hat{\beta}\left(\Omega(t_0),t-t_0\right),\qquad \text{for all } t\ge t_0. \tag{105}$$

Furthermore, (19) and (20) hold with $\delta^*(c)=\delta_1\left(\hat{\beta}\left(\psi_{\rm RoA}(c),0\right)\right)$, where δ_1 is defined in (36).

The idea of the proof of Theorem 3 is captured by the two plots in Fig. 8. The top plot in Fig. 8 depicts four possibilities (not an exhaustive list) that may arise with closed-loop solutions: 1) control never reaches the plant; 2) control reaches the plant but the state has already exited the region of attraction of the delay-free closed-loop system; 3) the state is within the region of attraction of the delay-free closed-loop system when the control reaches the plant but the subsequent transient leads the state outside the controller's feasibility region and, thus, to the reversal of the direction of the control signal; and 4) control reaches the plant while the state is within the region of attraction of the delay-free closed-loop system and the solution remains within the controller's feasibility region, so that the control signal is never retarded to earlier values, and the delay remains compensated for all subsequent times. The proof of Theorem 3 estimates a set of initial conditions from which all of the solutions belong to category 4).

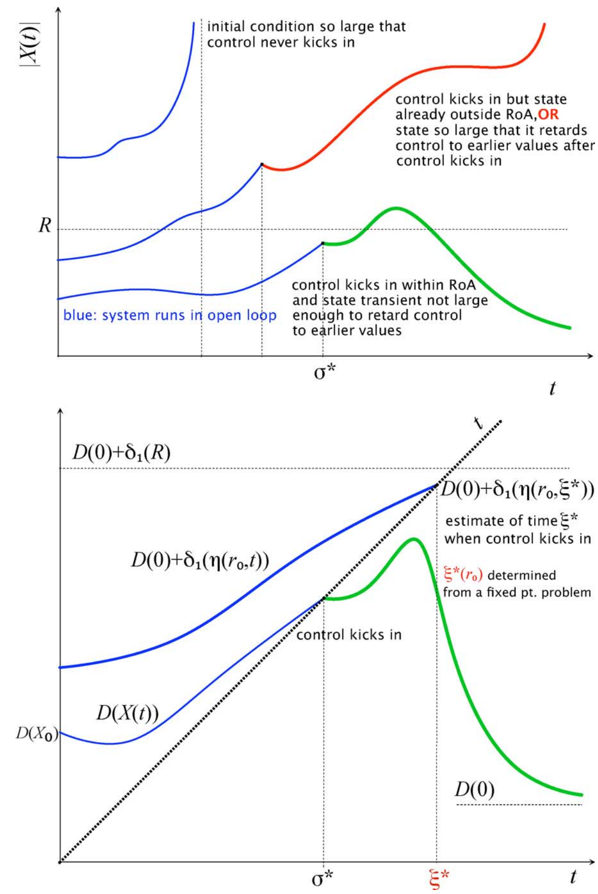

Fig. 8. Top: Possible transients of the state $X(t)$. Bottom: An example of a favorable transient of the delay $D\left(X(t)\right)$ together with an estimate of its upperbound as defined in (36) and (107).

The bottom plot in Fig. 8 depicts the strategy of the proof of Theorem 3. The exact time σ^* when the control reaches the plant is not known analytically. We find an upperbound $\xi^*\ge\sigma^*$ by using an upperbound $D(X)\le D(0)+\delta_1(|X|)$ on the delay and by estimating an upperbound on the open-loop solution $|X(t)|\le\eta(r_0,t-t_0)$, where $r_0:=\Omega(t_0)$. The upperbound ξ^* is then determined from the fixed-point problem $\xi^*-t_0=D(0)+\delta_1(\eta(r_0,\xi^*-t_0))$. The solution $\xi^*(r_0)$ to the fixed-point problem is a function of the size of the initial condition r_0. By reducing r_0 sufficiently, we can ensure that the control signal reaches the plant before $|X(t)|$ has exceeded R, namely, before $D(X(t))$ has exceeded the known bound $D(0)+\delta_1(R)$.

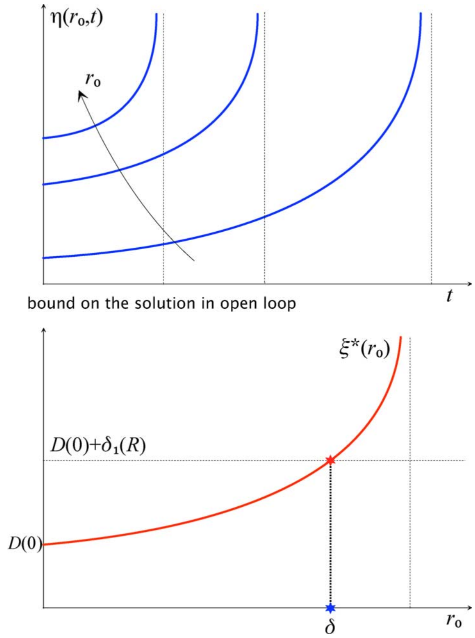

Fig. 9. Left: a representative form of the function $\eta(r_0, t)$ in Lemma 9. Right: Restricting the initial conditions in Lemma 11 to $r_0 \in [0, \delta]$ so that when the control kicks in, which is no later than $\xi^*(r_0)$, the delay is no greater than $D(0) + \delta_1(R)$ and, consequently, the plant state is no greater than R.

The actual proof of Theorem 3 consists of Lemmas 9–13, which are presented next.

Lemma 9 (A Bound on Open-Loop Solutions): For the plant (1), there exists a function $\eta(r, s) : \mathbb{B} \mapsto [0, \infty)$, where

$$\mathbb{B} = \left\{ (r, s) \in (\mathbb{R}_+)^2 : 0 \le s < \lim_{z \to \infty} \int_r^z \frac{d\mu}{\alpha_1(\mu)} \right\} \quad (106)$$

with the following properties:

a) $\eta(r, s)$ is increasing in both of its arguments r and s;

b) $\eta(r, s)$ is continuous in its domain of definition and, moreover, $\lim_{r \to 0} \eta(r, s) = 0$ uniformly in s;

such that for all $t \le \hat{t} = \min\left\{\sigma^*, t_0 + \lim_{z\to\infty} \int_{r_0}^z (d\mu/\alpha_1(\mu))\right\}$, where $\sigma^* = t_0 + D(X(\sigma^*))$, it holds that

$$|X(t)| \le \eta(r_0, t - t_0) \quad (107)$$

where $r_0 = \Omega(t_0) = |X(t_0)| + \sup_{t_0 - D(X(t_0)) \le \theta \le t_0} |U(\theta)|$.

Proof: The existence of the function η is proved by construction. A representative form of η is shown graphically in Fig. 9. Since for all $t \le \hat{t}$, the system runs in open loop, for all $t_0 \le t \le \hat{t}$ it holds that

$$\frac{d|X(t)|}{dt} \le \alpha_1\left(|X(t)| + \sup_{t_0 - D(X(t_0)) \le \theta \le t_0} |U(\theta)| \right). \quad (108)$$

Setting $y(t) = |X(t)| + \sup_{t_0 - D(X(t_0)) \le \theta \le t_0} |U(\theta)|$, we get

$$\dot{y}(t) \le \alpha_1(y(t)), \quad t_0 \le t \le \hat{t}. \quad (109)$$

Consider now the following ODE

$$\dot{z}(t) = \alpha_1(z(t)), \quad z(t_0) = y(t_0) \ge 0. \quad (110)$$

Define for any $a > 0$, the function $g(z)$ for all $z \in (0, \infty)$ as

$$g(z) = \int_a^z \frac{ds}{\alpha_1(s)}. \quad (111)$$

The function g is continuous and strictly increasing for all $z \in (0, \infty)$ since $g'(z) = 1/\alpha_1(z)$. The range of the function g is (b, d), where $b = \lim_{z \to 0^+} \int_a^z (ds/\alpha_1(s))$ and $d = \lim_{z\to\infty} \int_a^z (ds/\alpha_1(s))$. Note that b can be $-\infty$ and d can be $+\infty$. From (110), we obtain $\int_{z(t_0)}^{z(t)} (ds/\alpha_1(s)) = t - t_0$. By defining $z(t_0) = z_0$ (since from now on, we treat $z(t_0)$ as a paramater), we get $g(z(t)) = g(z_0) + t - t_0$ for all $t \ge t_0$ and $z_0 > 0$. Thus, $z(t) = g^{-1}(g(z_0) + t - t_0)$ for all $t \ge t_0$ and $z_0 > 0$. Define for all $s \ge 0$

$$\eta(r, s) = \begin{cases} g^{-1}(g(r) + s), & r > 0 \\ 0, & r = 0 \end{cases}. \quad (112)$$

From (112), we observe that

$$\eta : \left\{ (r, s) \in (\mathbb{R}_+)^2 : 0 \le g(r) + s < d \right\} = \mathbb{B} \mapsto [0, \infty)$$

and, moreover, η is increasing in both of its arguments since g is increasing. From its definition, η is continuous in its domain. To see this, first note that η is continuous in its domain with an exception maybe at the points $(0, s)$. Yet, since from (110) it holds that $z_0 = 0 \Rightarrow z(t) = 0,\ \forall t > t_0$, by continuity of $z(t)$ with respect to its initial conditions, $z_0 \to 0 \Rightarrow z(t) \to 0$ for all $t \in (t_0, l),\ l \to \infty$. Therefore, $\lim_{r\to 0} \eta(r, s) = 0 = \eta(0, s)$ uniformly in s. Using the comparison principle and (109), we get the statement of the lemma. ■

Lemma 10: (For Small Initial Conditions, The Upperbound on the Delay as a Function of the Time When the Control Kicks in is a Contraction): There exists a sufficiently small $\bar{r}_0$ such that for all $r_0 \in [0, \bar{r}_0]$, the mapping $T_{r_0}(s^*) = D(0) + \delta_1(\eta(r_0, s^*))$ is a contraction in $[D(0), D(0) + \delta_1(R)]$.

Proof: We start by setting $s^* = \xi^* - t_0$. Based on Lemma 9, since $L(r_0) = \lim_{z\to\infty} \int_{r_0}^z (d\mu/\alpha_1(\mu))$ satisfies $\lim_{r_0 \to 0} L(r_0) \to \infty$ there exists a sufficiently small $\hat{r}_0$ such that for all initial conditions $r_0 \in [0, \hat{r}_0]$ it holds that $\xi^* < t_0 + \lim_{z\to\infty} \int_{r_0}^z (d\mu/\alpha_1(\mu))$ and, hence, $\eta(r_0, \xi^* - t_0)$ is continuous in $(r_0, \xi^*) \in [0, \hat{r}_0] \times [t_0 + D(0), t_0 + D(0) + \delta_1(R)]$. Since $D \in C^1(\mathbb{R}^n; \mathbb{R}_+)$ (which allows us to choose $\delta_1 \in \mathcal{K}_\infty \cap C^1$) and by continuity of η and the fact that $\lim_{r\to 0} \eta(r, s) = 0$ uniformly in s, there exists a sufficiently small $r_0^* > 0$ such that for all $r_0 \in [0, r_0^*]$, $T_{r_0}[D(0), D(0) + \delta_1(R)] \subseteq [D(0), D(0) + \delta_1(R)]$ (e.g., $\delta_1(\eta(r_0^*, D(0) + \delta_1(R))) = \delta_1(R)$), that is, the mapping T_{r_0} maps the set $[D(0), D(0) + \delta_1(R)]$ into itself. Differentiating (112) with respect to s and using the fact that $g'(z) = 1/\alpha_1(z)$, we get $\partial\eta(r, s)/\partial s = 1/g'\left(g^{-1}(g(r) + s)\right) = \alpha_1(\eta(r_0, s))$. Hence, $dT_{r_0}(s^*)/ds^* = \delta_1'(\eta(r_0, s^*))\,\alpha_1(\eta(r_0, s^*))$, where $s^* = \xi^* - t_0$. Using the facts that $\delta_1 \in C^1$, $\alpha_1 \in \mathcal{K}_\infty$, and η is continuous with $\lim_{r\to 0} \eta(r, s) = 0$ uniformly in s, there exists a sufficiently small r_0^{**} such that for all $r_0 \in [0, r_0^{**}]$ and $\xi^* \in [t_0 + D(0), t_0 + D(0) + \delta_1(R)]$

$$\left| \frac{dT_{r_0}(s^*)}{ds^*} \right| = |\delta_1'(\eta(r_0, s^*))\,\alpha_1(\eta(r_0, s^*))| < 1 \quad (113)$$

where $s^* = \xi^* - t_0$. With [21, Lemma 3.1] $|T_{r_0}(s_1^*) - T_{r_0}(s_2^*)| \leq L|s_1^* - s_2^*|$, for all $s_1^*, s_2^* \in [D(0), D(0) + \delta_1(R)]$, where $s_1^* = \xi_1^* - t_0$, $s_2^* = \xi_2^* - t_0$, and L satisfies $L < 1$. Reference [21, Theor. B.1] guarantees that the mapping $T_{r_0}(\xi^* - t_0)$ is a contraction in $[D(0), D(0) + \delta_1(R)]$ for all $r_0 \in [0, \bar{r}_0]$, $\bar{r}_0 = \min\{\hat{r}_0, r_0^*, r_0^{**}\}$. ■

Lemma 11: (For Small Initial Conditions, the Plant State is Within the Delay-Free Region of Attraction When Control Kicks In): There exists a sufficiently small $\delta > 0$ such that for all solutions of the closed-loop system satisfying (9) for $0 < c < 1$ and for all initial conditions of the plant (1) satisfying

$$r_0 \leq \delta \tag{114}$$

there exists a $\sigma^* \geq t_0$ such that $\sigma^* = t_0 + D(X(\sigma^*))$ and, moreover

$$|X(\sigma^*)| \leq R. \tag{115}$$

That is, r_0 is inside the region of attraction of the controller.

Proof: As long as the solutions of the closed-loop system satisfy (9) for $0 < c < 1$, the function $\phi(t)$ is increasing, with $\phi(t_0) = t_0 - D(X(t_0))$. Hence, there exists a time σ^* at which $\sigma^* = t_0 + D(X(\sigma^*))$ and, moreover, based on (36), σ^* is finite when $|X(\sigma^*)|$ is finite. Consider now the fixed-point problem

$$T_{r_0}(\xi^* - t_0) = D(0) + \delta_1(\eta(r_0, \xi^* - t_0)) = \xi^* - t_0. \tag{116}$$

Since by Lemma 10, $T_{r_0}(\xi^* - t_0)$ is a contraction in $[D(0), D(0) + \delta_1(R)]$, the fixed-point problem (116) has a unique solution $\xi^*(r_0) \in [t_0 + D(0), t_0 + D(0) + \delta_1(R)]$. Differentiating (116) with respect to r_0, we get

$$\xi^{*'}(r_0) = \delta_1'(\eta(r_0, \xi^* - t_0))\,\alpha_1(\eta(r_0, \xi^* - t_0))\,\xi^{*'}(r_0) + \delta_1'(\eta(r_0, \xi^* - t_0))\frac{\alpha_1(\eta(r_0, \xi^* - t_0))}{\alpha_1(r_0)}.$$

Using (113), the fact that $\delta_1 \in \mathcal{K}_\infty \cap C^1$ and the fact that $\delta_1'(\eta(r_0, \xi^* - t_0))(\alpha_1(\eta(r_0, \xi^* - t_0))/\alpha_1(r_0)) > 0$ for all $r_0 > 0$, we have $\xi^{*'}(r_0) > 0$ and, hence, $\xi^*(r_0)$ is increasing. Since $\xi^{*'}(r_0)$ is continuous for $r_0 > 0$ we conclude that $\xi^*(r_0)$ is also continuous for $r_0 > 0$. Since $\lim_{r\to 0}\eta(r, s) = 0$ uniformly in s and $\delta_1 \in \mathcal{K}_\infty \cap C^1$, we have from (116) that given any $\epsilon > 0$, there exists a sufficiently small r_0 such that $|\xi^*(r_0) - t_0 - D(0)| < \epsilon$ and, hence, ξ^* is continuous also at zero. Since the function ξ^* is monotonically increasing, it is invertible. Denote $\delta = \min\left\{\bar{r}_0, {\xi^*}^{-1}(t_0 + D(0) + \delta_1(R))\right\}$, as depicted in Fig. 9 (right). Then, thanks to (116)

$$\eta(r_0, \xi^*(r_0) - t_0) = \delta_1^{-1}(\xi^*(r_0) - D(0) - t_0) \leq R \tag{117}$$

for all $r_0 \in [0, \delta]$. The proof is completed if $\xi^* \geq \sigma^*$ since then bound (115) holds. Assume that $\xi^* < \sigma^*$ and since $\phi(t)$ is increasing, we have $\phi(\xi^*) < \phi(\sigma^*)$. Since $\xi^* < \sigma^*$, using Lemma 9, we have $|X(t)| \leq \eta(r_0, t - t_0)$, for all $t \leq \xi^*$. Therefore

$$t - D(X(t)) \geq t - D(0) - \delta_1(\eta(r_0, t - t_0)), \quad t \leq \xi^*. \tag{118}$$

Consequently, $\phi(\xi^*) = \xi^* - D(X(\xi^*)) \geq t_0 = \phi(\sigma^*)$, which contradicts the assumption. Thus, $\xi^* \geq \sigma^*$. Using (117) and the fact that the function $\eta(r, s)$ is increasing in s, we get (115). ■

Lemma 12 (Stability Estimate): There exists a class $\mathcal{K}_\infty$ function $\hat{\alpha}$ such that for all solutions of the closed-loop system (1), (3)–(5) that satisfy (9) for $0 < c < 1$, it holds that

$$\Omega(t) \leq 2\beta^*(\eta(r_0, \sigma^* - t_0), \max\{0, t - \sigma^*\}) + \hat{\alpha}(\beta^*(\eta(r_0, \sigma^* - t_0), \max\{0, t - \sigma^*\})) \tag{119}$$

for all $t \geq t_0$, where Ω is defined in (17).

Proof: Under Assumption 5 and using Lemma 11, we get

$$|X(t)| \leq \beta^*(|X(\sigma^*)|, t - \sigma^*), \quad t \geq \sigma^*. \tag{120}$$

Using Lemma 9 and the fact that without loss of generality, $\beta^*(s, 0) \geq s$, with (120), we get for all $t \geq t_0$

$$|X(t)| \leq \beta^*(\eta(r_0, \sigma^* - t_0), \max\{0, t - \sigma^*\}). \tag{121}$$

Since $U(\theta) = \kappa(\sigma(\theta), X(\sigma(\theta)))$ for all $\theta \geq t_0$, with (120) and (15) we get for all $t \geq \sigma^*$

$$\sup_{t - D(X(t)) \leq \theta \leq t} |U(\theta)| \leq \hat{\alpha}(\beta^*(|X(\sigma^*)|, t - \sigma^*)). \tag{122}$$

Moreover, for $t \leq \sigma^*$, we have

$$\sup_{t - D(X(t)) \leq \theta \leq t} |U(\theta)| \leq \sup_{t - D(X(t)) \leq \theta \leq t_0} |U(\theta)| + \sup_{t_0 \leq \theta \leq t} |U(\theta)|. \tag{123}$$

Hence, for all $t \leq \sigma^*$

$$\sup_{t - D(X(t)) \leq \theta \leq t} |U(\theta)| \leq \sup_{t_0 - D(X(t_0)) \leq \theta \leq t_0} |U(\theta)| + \hat{\alpha}(\beta^*(|X(\sigma^*)|, 0)). \tag{124}$$

Combining bounds (121), (122), and (124), we get the statement of the lemma. ■

Lemma 13: (Ball of Initial Conditions That Guarantees Controller Feasibility): There exists a class $\mathcal{K}$ function $\mu : [0, 1) \mapsto [0, \infty)$ such that for all initial conditions of the plant that satisfy (104) with $\psi_{\mathrm{RoA}}(c) \leq \min\{\delta, \mu(c)\}$, the solutions of the closed-loop system (1), (3)–(5) satisfy (9) for $0 < c < 1$.

Proof: Using (43), we have that $t_0 \leq y \leq \sigma^*$ for all $t_0 - D(X(t_0)) \leq \theta \leq t_0$. Comparing (44) with (1) and using Lemma 9, we get for all $t_0 - D(X(t_0)) \leq \theta \leq t_0$

$$|P(\theta)| \leq \eta(r_0, \sigma^* - t_0). \tag{125}$$

By noting that $P(\theta) = X(\sigma(\theta))$ for all $\theta \geq t_0$, with the help of bound (121), we have that

$$|P(\theta)| \leq \beta^*(\eta(r_0, \sigma^* - t_0), 0), \quad \theta \geq t_0. \tag{126}$$

Therefore, using the fact that $\beta^*(s, 0) \geq s$ from (125) and (126), we get for all $\theta \geq t_0 - D(X(t_0))$ that

$$|P(\theta)| \leq \beta^*(\eta(r_0, \sigma^* - t_0), 0). \tag{127}$$

Using (122) and (124), we get for all $t \geq t_0$ that

$$\sup_{t-D(X(t))\leq\theta\leq t} |U(\theta)| \leq r_0 + \hat{\alpha}\left(\beta^*\left(\eta\left(r_0, \sigma^* - t_0\right), 0\right)\right). \tag{128}$$

The solutions of the system lie inside the feasibility region as long as (57) holds for all $t - D(X(t)) \leq \theta \leq t$ and for all $t \geq t_0$. With the help of the fact that $\sigma^* \leq \xi^*(r_0)$ and using bounds (125), (127), and (128), (57) holds if

$$\begin{aligned} F(r_0) := &\left(|\nabla D(0)| + \delta_2\left(\beta^*\left(\eta\left(r_0, \xi^*(r_0) - t_0\right), 0\right)\right)\right) \\ &\times \alpha_1\big(\beta^*\left(\eta\left(r_0, \xi^*(r_0) - t_0\right), 0\right) \\ &+ r_0 + \hat{\alpha}(\beta^*(\eta\left(r_0, \xi^*(r_0) - t_0\right), 0))\big) \leq c. \end{aligned} \tag{129}$$

Condition (129) is satisfied if (104) holds with any class $\mathcal{K}$ choice $\psi_{\mathrm{RoA}}(c) \leq \min\{\delta, \mu(c)\}$, where $\mu(c) = \sup_{\mu^* \in \mathbb{A}} \mu^*$ and $\mathbb{A} = \{\mu^* \in \mathbb{R}_+ : F(\mu^*) \leq c\}$. Using the fact that $\xi^*(\cdot)$ is a monotonically increasing function, $F(\cdot)$ is also increasing with $F(0) = 0$. Hence, $\mu(c) = \sup_{\mu^* \in \mathbb{A}} \mu^*$ is continuous and increasing, with $\mu(0) = 0$. ■

Proof of Theorem 3: Bound (105) follows from Lemma 12 and the fact that $\xi^*(r)$ is increasing, $\sigma^* \leq \xi^*$ and $D(0) + \delta_1\left(\eta\left(r_0, \xi^*(r) - t_0\right)\right) = \xi^*(r) - t_0$, with

$$\begin{aligned} \hat{\beta}(r, t - t_0) = &\, 2\beta^*\left(\eta\left(r, \xi^*(r) - t_0\right), \max\{0, t - \xi^*(r)\}\right) \\ &+ \hat{\alpha}\left(\beta^*\left(\eta\left(r, \xi^*(r) - t_0\right), \max\{0, t - \xi^*(r)\}\right)\right). \end{aligned} \tag{130}$$

The rest of the results follow as in the proof of Theorem 1. ■

1) Example 3: We consider the scalar system

$$\dot{X}(t) = X(t)^4 + 2X(t)^5 + \left(X(t)^2 + X(t)^3\right) U\left(t - X(t)^2\right). \tag{131}$$

The origin of (131) is neither locally exponentially stabilizable nor globally asymptotically stabilizable (because it is not reachable for $X(0) < -1$). In the delay-free case, the controller $U(t) = -X(t)$ yields a closed-loop system $\dot{X}(t) = -X(t)^3 + 2X(t)^5$, which is locally asymptotically stable, with $R = 1/\sqrt{2}$. We assume for simplicity of the analysis that $U(\theta) = 0$ for all $\theta \leq 0$. Denoting

$$y_0 = 8\log\left(\left|X(0) + \frac{1}{2}\right|\right) - 8\log(|X(0)|) - \frac{4X(0)^2 - X(0) + \frac{1}{3}}{X(0)^3}$$

the controller "kicks in" at the time $\sigma^* = (X^*)^2$, where X^* satisfies

$$8\log\left(\left|X^* + \frac{1}{2}\right|\right) - 8\log\left(|X^*|\right) - \frac{4\left(X^*\right)^2 - X^* + \frac{1}{3}}{\left(X^*\right)^3} - y_0 = \left(X^*\right)^2. \tag{132}$$

Let $X(0) = 0.543$. Solving (132), we obtain $\sigma^* = 0.46$ and $X^* = \sqrt{\sigma^*} = 0.678$, which is almost at $R = 1/\sqrt{2}$. The predictor controller is $U(t) = -P(t)$

$$P(t) = X(t) + \int_{t-X(t)^2}^{t} \frac{f\left(P(\theta), U(\theta)\right) d\theta}{1 - 2P(\theta) f\left(P(\theta), U(\theta)\right)} \tag{133}$$

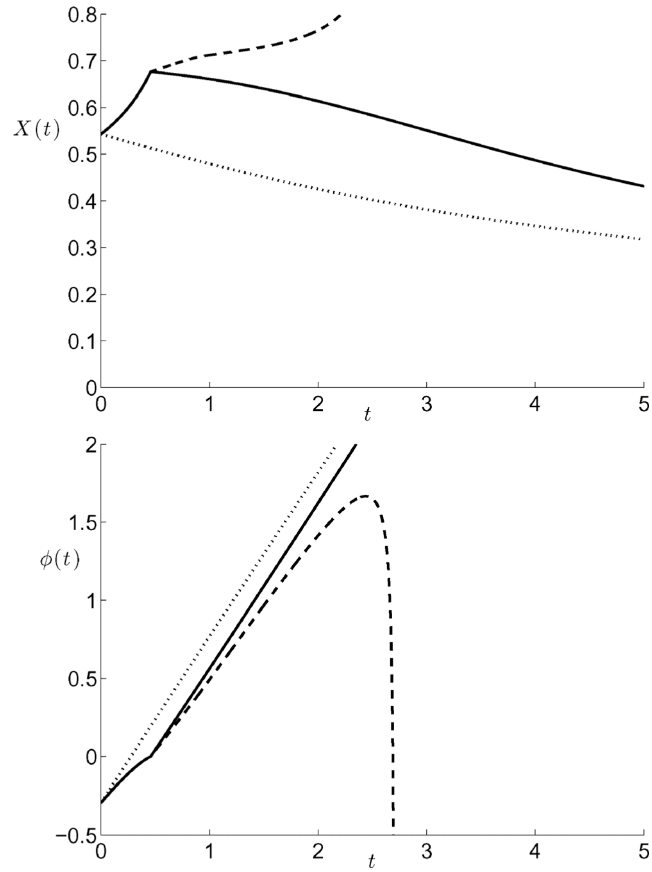

Fig. 10. State $X(t)$, the control effort $U(t)$, and the function $\phi(t) = t - X(t)^2$ of system (131) with the delay-compensating controller (solid line), the uncompensated controller (dashed line), and the nominal controller for a system without delay (dotted line) for $X(0) = 0.543$ and $U(s) = 0$ for all $s \leq 0$.

where

$$f\left(P(\theta), U(\theta)\right) = P(\theta)^4 + 2P(\theta)^5 + \left(P(\theta)^2 + P(\theta)^3\right) U(\theta).$$

In Fig. 10, we show that the predictor feedback achieves local asymptotic stabilization.

VII. Conclusions

The paper's key design idea is how to define the predictor state (4). The gradient-of-delay term in the denominator of (4) is the result of a change in the time variable, which allows the predictor to be defined using an integral from the known delayed time $\phi(t) = t - D(X(t))$ until present time t, rather than an integral from the present time t until the unknown prediction time $\sigma(t) = t + D(P(t))$.

Though the stability results in this paper are not global, the size of the delay is not limited. By examining the estimates in detail, the reader can observe that when the delay $D(X)|_{X=0}$ is large, namely, when the system is regulated to an equilibrium where the delay is necessarily large, the stability estimates dictate that the initial conditions of the state and the input be small. However, no restrictions on $D(X)|_{X=0}$ are imposed. A tradeoff exists between the size of the state-dependent delay and the achievable region of attraction in closed loop.

With the Lyapunov construction in the case of forward-complete systems, one can, in principle, pursue a robustness study when the uncertainties on the delay and its gradient are restricted (which can be done in some cases by shrinking the state X) and when the uncertainty in the parameters is sufficiently small. Paper [23], where robustness to delay mismatch is proven for the case of linear plants with constant delay, constitutes a starting point.

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