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Adaptive stabilization of LTI systems with distributed input delay

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SUMMARY

We solve stabilization problems of LTI systems with unknown parameters and distributed input delay. The key challenge is that the infinite-dimensional input dynamics are distributed, which makes traditional infinite-dimensional backstepping inapplicable. We resolve this challenge by employing backstepping–forwarding transformations of the finite-dimensional state of the plant and of the infinite-dimensional actuator state. These transformations enable us to design Lyapunov-based update laws. We also design an adaptive controller for rejecting a constant disturbance in the input of the LTI plant, in the case where the parameters of the plant are known. Copyright © 2012 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Infinite-dimensional backstepping has been very successful in designing stabilizing controllers for systems that have infinite-dimensional actuator and sensor dynamics [1]. Such actuator or sensor dynamics arise in various engineering applications such as population dynamics, traffic flows, chemical reactors, and heat exchangers [1].

Although there is rich literature on the compensation of input and state delays [2–13], and on the adaptive control through backstepping [14–16], few papers are dealing with the adaptive control of time-delay systems. In [17–19], the finite spectrum assignment technique is employed to design adaptive controllers for linear plants with unknown parameters and known, lumped input delays. References [20, 21] dealt with the adaptive control of linear and nonlinear systems, respectively, which have lumped state delays. The problem of stabilization of linear systems with unknown lumped input delays and unknown plant parameters has been solved recently in [22, 23], using tools coming from the adaptive control of parabolic PDEs [24–26].

However, the results from [22, 23] are not applicable to the case of distributed input delay, as the finite-dimensional state of the plant and the infinite-dimensional actuator state are not in the strict-feedback form [27]. Yet, compensation of distributed infinite-dimensional input dynamics of convection type is achieved by combining infinite-dimensional backstepping and infinite-dimensional forwarding [27, 28]. In the present case, we generalize the backstepping–forwarding transformations from [27, 28] to the case where the parameters of the plant are unknown and to the case where the parameters of the plant are known but there is a matched, constant disturbance in the input. One of the main challenges of this generalization in the former case is that one has to deal, in the case of the B matrix, with a vector of unknown functions, rather than just with a

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vector of unknown parameters. We resolve this challenge by constructing a Lyapunov functional with normalization and by employing an update law using projection on a projection set, which can be either spherical or an infinite-dimensional hyper-rectangle. In addition, the gain kernels of these transformations are time varying, as they incorporate the estimations of the unknown parameters, and hence, various technical difficulties arise when one proves that these kernels are bounded (which we need in our Lyapunov analysis). In the latter case, one has to appropriately incorporate into the backstepping–forwarding transformations the estimation of the unknown disturbance, in order to account for its effect.

We introduce backstepping–forwarding transformations, of certainty equivalence type, of the finite-dimensional plant state and of the infinite-dimensional actuator state that transform the system to a 'target system'. By constructing a Lyapunov function for the target system, we design update laws for the parameters of the plant, which in the case of the input matrix B is infinite dimensional. With the help of the available Lyapunov function, we prove stability and regulation of the closed-loop system (Section 2).

For linear systems with distributed input delay and assuming that the plant's parameters are known, we design a control law that stabilizes the closed-loop system and achieves compensation of a constant disturbance in the input of the plant. With the available infinite-dimensional backstepping–forwarding transformations and treating the disturbance as an unknown parameter, we design an update law with the aid of a Lyapunov function that we construct (Section 3).

Notation: The set $\mathbb{R}^n$ is the set of all real vectors of dimension n. For a vector $x \in \mathbb{R}^n$, we denote by $|x|$ its Euclidean norm. The set $\mathbb{R}_+$ denotes the set of non-negative real numbers. With $f \in C^l(A;\Omega)$, we denote a function that is defined in $A \subseteq \mathbb{R}^n$, and it takes values in $\Omega \subseteq \mathbb{R}^n$ and has continuous derivatives of order l on A.

2. ADAPTIVE CONTROL FOR LTI PLANTS WITH UNKNOWN PARAMETERS

We consider the system

$$\dot{X}(t) = AX(t) + \int_0^D B(D-\sigma)U(t-\sigma)\mathrm{d}\sigma, \tag{1}$$

where $X \in \mathbb{R}^n$ is the state of plant, $U \in \mathbb{R}$ is the input, and $D > 0$ is the delay. For notational simplicity, we assume that our system is single input. However, the results of this section can be extended to the multi-input case, when the delays are the same in each individual input channel. We are concerned here with the case where the matrices $B(x)$, $x \in [0, D]$, and A are unknown and of the form

$$A = A_0 + \sum_{i=1}^{p} \theta_i A_i \tag{2}$$

$$B(x) = B_0(x) + \sum_{i=1}^{p} b_i(x) B_i, \tag{3}$$

where the $b_i(x)$, $i = 1, 2, \ldots, p$ are unknown, scalar continuous functions of x, and θ_i, $i = 1, 2, \ldots, p$ are unknown constants.

In order to help better understand the structure of system (1)–(3), we derive its transfer function, namely $G(s)$, from the input $U(t)$ to the state $X(t)$. We first re-write system (1) in the following equivalent form using transport PDE representation for the actuator state $U(\sigma)$, $\sigma \in [t-D, t]$ as

$$\dot{X}(t) = AX(t) + \int_0^D B(x)u(x,t)\mathrm{d}x \tag{4}$$

$$u_t(x,t) = u_x(x,t), \quad x \in [0, D] \tag{5}$$

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$$u(D,t) = U(t). \tag{6}$$

Taking the Laplace transform of (5), we obtain the following boundary value problem with respect to the spatial variable x

$$su(s,x) = u'(s,x) \tag{7}$$

$$u(s,D) = U(s). \tag{8}$$

Solving the preceding boundary value problem, we obtain

$$u(s,x) = e^{s(x-D)}U(s). \tag{9}$$

Taking the Laplace transform of (4) and using (9), we obtain

$$G(s) = (sI - A)^{-1} \int_0^D B(x)e^{s(x-D)}\mathrm{d}x. \tag{10}$$

Our adaptive controller is based on infinite-dimensional update laws for the estimation of the unknown functions $b_i(x)$, $i = 1, 2, \ldots, p$ for all $x \in [0, D]$ and on finite-dimensional update laws for the estimation of the constant parameters θ_i, $i = 1, 2, \ldots, p$. We employ the update laws using projector operators (see [26] for the use of projector operators in PDEs). Consequently, we make the following assumption.

Assumption 1
There exist known constants $\underline{\theta}_i$, $\overline{\theta}_i$, ρ_i and known continuous functions $b_i^{\mathrm{n}}(x)$, $i = 1, 2, \ldots, p$ such that

$$\int_0^D \left(b_i(x) - b_i^{\mathrm{n}}(x)\right)^2 \mathrm{d}x \leqslant \rho_i \tag{11}$$

$$\underline{\theta}_i \leqslant \theta_i \leqslant \overline{\theta}_i, \tag{12}$$

for all $i = 1, 2, \cdots, p$.

Furthermore, we have to make an assumption regarding the controllability of a specific pair of matrices (this fact becomes clear later on) for all $B(x)$ such that $\int_0^D \left(b_i(x) - b_i^{\mathrm{n}}(x)\right)^2 \mathrm{d}x \leqslant \rho_i$, $i = 1, 2, \cdots, p$ and for all A such that $\underline{\theta}_i \leqslant \theta_i \leqslant \overline{\theta}_i$, $i = 1, 2, \cdots, p$. We thus make the following assumption.

Assumption 2
We assume that the pair $\left(A, \int_0^D e^{-A(D-x)}B(x)\mathrm{d}x\right)$ is uniformly completely controllable for all $\underline{\theta}_i \leqslant \theta_i \leqslant \overline{\theta}_i$, $i = 1, 2, \cdots, p$ and for all $\int_0^D \left(b_i(x) - b_i^{\mathrm{n}}(x)\right)^2 \mathrm{d}x \leqslant \rho_i$, $i = 1, 2, \cdots, p$, and that there exist a vector valued function $K(\cdot) : C^1(\Lambda; R^n)$ and matrices $P(\cdot) : C^1\left(\Lambda; \mathbb{R}_+^{n\times n}\right)$, $Q(\cdot) : C\left(\Lambda; \mathbb{R}_+^{n\times n}\right)$, where

$$\Lambda = \left\{ (\beta, \theta) \in \mathbb{R}^{n+p} \mid \beta = \int_0^D e^{-A(D-x)}B(x)\mathrm{d}x, \right.$$
$$\left. \text{for all } \int_0^D \left(b_i(x) - b_i^{\mathrm{n}}(x)\right)^2 \mathrm{d}x \leqslant \rho_i \text{ and } \underline{\theta}_i \leqslant \theta_i \leqslant \overline{\theta}_i, i = 1, 2, \cdots, p \right\}, \tag{13}$$

symmetric and positive definite such that

$$\left(A+\int_0^D e^{-A(D-x)}B(x)\mathrm{d}xK(\beta,\theta)\right)^{\mathrm{T}}P(\beta,\theta)$$
$$+P(\beta,\theta)\left(A+\int_0^D e^{-A(D-x)}B(x)\mathrm{d}xK(\beta,\theta)\right)=-Q(\beta,\theta). \tag{14}$$

We make next our final assumption that is used in our stability analysis (see also [22]).

Assumption 3
The quantities $\underline{\lambda}=\inf_{(\beta,\theta)\in\Lambda}\min\{\lambda_{\min}(Q(\beta,\theta)),\lambda_{\min}(P(\beta,\theta))\}$ and $\overline{\lambda}=\sup_{(\beta,\theta)\in\Lambda}\lambda_{\max}(P(\beta,\theta))$ exist and are known.

The controller for system (1) is

$$U(t)=K\left(\hat{\beta},\hat{\theta}\right)\hat{Z}(t), \tag{15}$$

where

$$\hat{Z}(t)=X(t)+\int_0^D\int_0^x e^{-\hat{A}(t)(x-y)}\hat{B}(y,t)\mathrm{d}yu(x,t)\mathrm{d}x. \tag{16}$$

The update laws are given by

$$\dot{\hat{b}}_i(x,t)=\gamma_b\mathrm{Proj}\left\{\tau_{b_i},\hat{b}_i,b_i^{\mathrm{n}},\rho_i\right\}(x) \tag{17}$$

$$\dot{\hat{\theta}}_i(t)=\gamma_\theta\mathrm{Proj}_{[\underline{\theta}_i,\overline{\theta}_i]}\left\{\tau_{\theta_i},\hat{\theta}_i\right\}, \tag{18}$$

where the projector operators are defined as

$$\mathrm{Proj}\left\{\tau,\hat{\xi},\xi^{\mathrm{n}},\rho\right\}(x)=\begin{cases}\tau(x)-\left(\hat{\xi}(x)-\xi^{\mathrm{n}}(x)\right)\frac{\langle\hat{\xi}-\xi^{\mathrm{n}},\tau\rangle}{\|\hat{\xi}-\xi^{n}\|^2}, & \text{if } \|\hat{\xi}-\xi^{\mathrm{n}}\|^2=\rho \text{ and}\\ & \langle\hat{\xi}-\xi^{\mathrm{n}},\tau\rangle>0\\ \tau(x), & \text{otherwise}\end{cases} \tag{19}$$

where

$$\|\hat{\xi}-\xi^{\mathrm{n}}\|^2=\int_0^D\left(\hat{\xi}(x)-\xi^{\mathrm{n}}(x)\right)^2\mathrm{d}x \tag{20}$$

$$\left\langle\hat{\xi}-\xi^{\mathrm{n}},\tau\right\rangle=\int_0^D\left(\hat{\xi}(x)-\xi^{\mathrm{n}}(x)\right)\tau(x)\mathrm{d}x, \tag{21}$$

and

$$\mathrm{Proj}_{[\underline{r},\overline{r}]}\left\{\tau,\hat{\zeta}\right\}=\begin{cases}0, & \text{if } \hat{\zeta}=\underline{r} \text{ and } \tau<0\\ 0, & \text{if } \hat{\zeta}=\overline{r} \text{ and } \tau>0\\ \tau, & \text{otherwise}\end{cases} \tag{22}$$

where

$$\tau_{b_i}(x)=\frac{\hat{Z}(t)^{\mathrm{T}}P\left(\hat{\beta},\hat{\theta}\right)-\int_0^D(1+y)g(y,t)w(y,t)\mathrm{d}y}{1+\Xi(t)}B_iu(x,t) \tag{23}$$

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$$\tau_{\theta_i} = \frac{\int_0^D (1+x)w(x,t)g(x,t)\mathrm{d}x - \hat{Z}(t)^{\mathrm{T}} P\left(\hat{\beta},\hat{\theta}\right)}{1+\Xi(t)} \times A_i \left(\int_0^D \int_0^x e^{-\hat{A}(x-y)} \hat{B}(y,t)\mathrm{d}y u(x,t)\mathrm{d}x - \hat{Z}(t)\right) \tag{24}$$

$$w(x,t) = u(x,t) - g(x,t)\hat{Z}(t) \tag{25}$$

$$\Xi(t) = \hat{Z}(t)^{\mathrm{T}} P\left(\hat{\beta},\hat{\theta}\right)\hat{Z}(t) + \int_0^D (1+x)w(x,t)^2 \mathrm{d}x \tag{26}$$

$$g_t(x,t) = -g(x,t)A_{\mathrm{cl}}\left(\hat{\beta}(t),\hat{\theta}(t)\right) + g_x(x,t) \tag{27}$$

$$g(D,t) = K\left(\hat{\beta}(t),\hat{\theta}(t)\right) \tag{28}$$

$$A_{\mathrm{cl}}\left(\hat{\beta}(t),\hat{\theta}(t)\right) = \hat{A} + \hat{\beta}(t)K\left(\hat{\beta}(t),\hat{\theta}(t)\right) \tag{29}$$

$$\hat{\beta}(t) = \int_0^D e^{-\hat{A}(D-x)}\hat{B}(x,t)\mathrm{d}x. \tag{30}$$

We now state our main result.

Theorem 1

Consider the closed-loop systems consisting of the plant (1) together with the adaptive controller (15)–(30). Let Assumptions 1–3 be satisfied and choose γ_θ and γ_b such that

$$\gamma_\theta + \gamma_b < \frac{\min\{\underline{\lambda}, 1\}^2}{M}, \tag{31}$$

where M is a sufficiently large constant. Then there exist positive constants R and ρ such that

$$\Omega(t) \leqslant R\left(e^{\rho\Omega(0)} - 1\right), \tag{32}$$

where

$$\Omega(t) = |X(t)|^2 + \|u(t)\|^2 + \left\|\tilde{b}(t)\right\|^2 + \left|\tilde{\theta}(t)\right|^2, \tag{33}$$

and

$$\|u(t)\|^2 = \int_0^D u^2(x,t)\mathrm{d}x \tag{34}$$

$$\left\|\tilde{b}(t)\right\|^2 = \int_0^D \left|\tilde{b}(x,t)\right|^2 \mathrm{d}x \tag{35}$$

$$\tilde{b}(x,t) = b(x) - \hat{b}(x,t) \tag{36}$$

$$\tilde{\theta}(t) = \theta - \hat{\theta}(t). \tag{37}$$

Furthermore,

$$\lim_{t\to\infty} X(t) = 0 \tag{38}$$

$$\lim_{t\to\infty} U(t) = 0. \tag{39}$$

We start proving the preceding theorem by first noting that relations (16) and (25) define two transformations, namely $\hat{Z}(t)$ and $w(x,t)$, of the form

$$(X(t), u(x,t)) \to \left(\hat{Z}(t), w(x,t)\right). \tag{40}$$

With these transformations, system (4)–(6) is mapped to the target system. Differentiating (16) with respect to time, we obtain that

$$\begin{aligned}\dot{\hat{Z}}(t) &= \hat{A}\hat{Z}(t) + \tilde{A}\hat{Z}(t) + \int_0^D e^{-\hat{A}(D-x)}\hat{B}(x,t)\mathrm{d}xU(t) + \int_0^D \tilde{B}(x,t)u(x,t)\mathrm{d}x \\ &\quad + \int_0^D \int_0^x e^{-\hat{A}(x-y)}\dot{\hat{B}}(y,t)\mathrm{d}yu(x,t)\mathrm{d}x - \tilde{A}\int_0^D \int_0^x e^{-\hat{A}(x-y)}\hat{B}(y,t)\mathrm{d}y \\ &\quad \times u(x,t)\mathrm{d}x - \dot{\hat{A}}\int_0^D \int_0^x (x-y)e^{-\hat{A}(x-y)}\hat{B}(y,t)\mathrm{d}yu(x,t)\mathrm{d}x,\end{aligned} \tag{41}$$

where we also used (4)–(6). Using (15), we obtain

$$\begin{aligned}\dot{\hat{Z}}(t) &= A_{\mathrm{cl}}\left(\hat{\beta},\hat{\theta}\right)\hat{Z}(t) + \tilde{A}\hat{Z}(t) + \int_0^D \tilde{B}(x,t)u(x,t)\mathrm{d}x \\ &\quad + \int_0^D \int_0^x e^{-\hat{A}(x-y)}\dot{\hat{B}}(y,t)\mathrm{d}yu(x,t)\mathrm{d}x - \tilde{A}\int_0^D \int_0^x e^{-\hat{A}(x-y)}\hat{B}(y,t)\mathrm{d}y \\ &\quad \times u(x,t)\mathrm{d}x - \dot{\hat{A}}\int_0^D \int_0^x (x-y)e^{-\hat{A}(x-y)}\hat{B}(y,t)\mathrm{d}yu(x,t)\mathrm{d}x.\end{aligned} \tag{42}$$

Moreover, differentiating (25) with respect to time and with respect to the spatial variable x, we obtain

$$\begin{aligned}w_t(x,t) &= u_x(x,t) - \left(g_t(x,t) + g(x,t)A_{\mathrm{cl}}\left(\hat{\beta},\hat{\theta}\right)\right)\hat{Z}(t) - g(x,t)\left(\int_0^D \tilde{B}(x,t)u(x,t)\mathrm{d}x\right. \\ &\quad \left. + \int_0^D \int_0^x e^{-\hat{A}(x-y)}\dot{\hat{B}}(y,t)\mathrm{d}yu(x,t)\mathrm{d}x\right) - g(x,t)\left(-\tilde{A}\int_0^D \int_0^x e^{-\hat{A}(x-y)}\hat{B}(y,t)\right. \\ &\quad \left. \times \mathrm{d}yu(x,t)\mathrm{d}x - \dot{\hat{A}}\int_0^D \int_0^x (x-y)e^{-\hat{A}(x-y)}\hat{B}(y,t)\mathrm{d}yu(x,t)\mathrm{d}x + \tilde{A}\hat{Z}(t)\right),\end{aligned} \tag{43}$$

and

$$w_x(x,t) = u_x(x,t) - g_x(x,t)\hat{Z}(t), \tag{44}$$

respectively. As $g(x,t)$ satisfies (27)–(28), we have that

$$\begin{aligned}w_t(x,t) &= w_x(x,t) \\ &\quad - g(x,t)\left(\int_0^D \tilde{B}(x,t)u(x,t)\mathrm{d}x + \int_0^D \int_0^x e^{-\hat{A}(x-y)}\dot{\hat{B}}(y,t)\mathrm{d}yu(x,t)\mathrm{d}x\right) \\ &\quad + g(x,t)\left(\tilde{A}\int_0^D \int_0^x e^{-\hat{A}(x-y)}\hat{B}(y,t)\mathrm{d}yu(x,t)\mathrm{d}x\right. \\ &\quad \left. + \dot{\hat{A}}\int_0^D \int_0^x (x-y)e^{-\hat{A}(x-y)}\hat{B}(y,t)\mathrm{d}yu(x,t)\mathrm{d}x - \tilde{A}\hat{Z}(t)\right)\end{aligned} \tag{45}$$

$$w(D,t) = 0. \tag{46}$$

From relation (16), one should notice that $\hat{Z}$ is given explicitly in terms of (X,u). Substituting this explicit relation for $\hat{Z}$ (in terms of (X,u)) into (25), one can observe that $w(x,t)$ is also written in terms of (X,u). Hence, the state $\left(\hat{Z}, w\right)$ defines a transformation of the state (X,u). We define now the inverse of the transformation in (25) as

$$u(x,t) = w(x,t) + g(x,t)\hat{Z}(t). \tag{47}$$

Consequently, we can write (45)–(46) only in terms of the transformed variables, that is, in terms of $w(x,t)$ and $\hat{Z}(t)$. Using (16) and (47), we define the inverse transformation of $\hat{Z}(t)$ as

$$\begin{aligned} X(t) = &\left(I - \int_0^D \int_0^x e^{-\hat{A}(x-y)}\hat{B}(y,t)\mathrm{d}y g(x,t)\mathrm{d}x\right)\hat{Z}(t) \\ &- \int_0^D \int_0^x e^{-\hat{A}(x-y)}\hat{B}(y,t)\mathrm{d}y w(x,t)\mathrm{d}x. \end{aligned} \tag{48}$$

Thus now, system (4)–(6) with state (X,u) is mapped into the target system, which is composed of relations (42) and (45)–(46) and has state $\left(\hat{Z}, w\right)$. The target system is obtained from the original system through a direct transformation of the form $(X,u) \to \left(\hat{Z}, w\right)$, which is defined in relations (16) and (25). With the inverse of this transformation, which is of the form $\left(\hat{Z}, w\right) \to (X,u)$ and is given in (47) and (48), one obtains the original system from the target system.

We state now a lemma that is concerned with the uniform boundness of the original and transformed variables.

Lemma 1
There exist constants M_u, M_w, M_X, and $M_{\hat{Z}}$ such that

$$\|u(t)\|^2 \leqslant M_u\left(\|w(t)\|^2 + |\hat{Z}(t)|^2\right) \tag{49}$$

$$|X(t)|^2 \leqslant M_X\left(|\hat{Z}(t)|^2 + \|w(t)\|^2\right) \tag{50}$$

$$\|w(t)\|^2 \leqslant M_w\left(\|u(t)\|^2 + |X(t)|^2\right) \tag{51}$$

$$|\hat{Z}(t)|^2 \leqslant M_{\hat{Z}}\left(|X(t)|^2 + \|u(t)\|^2\right). \tag{52}$$

Proof
First, observe that the signals $A\left(\hat{\theta}\right)$, $K\left(\hat{\beta},\hat{\theta}\right)$, and $P\left(\hat{\beta},\hat{\theta}\right)$ are continuously differentiable with respect to $\left(\hat{\beta},\hat{\theta}\right)$. Hence, as $\left(\hat{\beta},\hat{\theta}\right)$ is uniformly bounded, the signals $A\left(\hat{\theta}\right)$, $K\left(\hat{\beta},\hat{\theta}\right)$, $P\left(\hat{\beta},\hat{\theta}\right)$, and their derivatives are also uniformly bounded. Denote by M_A, M_K, M_P the bounds of $A\left(\hat{\theta}\right)$, $K\left(\hat{\beta},\hat{\theta}\right)$, $P\left(\hat{\beta},\hat{\theta}\right)$, respectively, and with M_A', M_K', M_P' the bound of their derivatives. We first prove that g satisfying (27)–(28) is uniformly bounded, that is, there exists a $\mu > 0$ such that

$$\int_0^D |g(x,t)|^2\mathrm{d}x \leqslant \mu^2, \quad \text{for all } t \geqslant 0. \tag{53}$$

Taking the time derivative of

$$V(t) = \int_0^D e^{mx}|g(x,t)|^2\mathrm{d}x, \tag{54}$$

where $m > 0$ is arbitrary, and using (27)–(30), we obtain that

$$\dot{V}(t) \leqslant e^{mD} M_K^2 - \left(m - 2\left(M_A + e^{M_A D}\sqrt{DM_2} M_K\right)\right) V(t), \tag{55}$$

where we also used integration by parts and the fact that A_{cl} in (29) satisfies $|A_{\mathrm{cl}}| \leqslant M_A + e^{M_A D}\sqrt{DM_2} M_K$, where

$$M_2 = (p+1)\left(D|B_0|^2 + 2\sum_{i=1}^{p} |B_i|^2 \left(\rho_i + \|b_i^n\|^2\right)\right). \tag{56}$$

Choosing $m = 2\left(M_A + e^{M_A D}\sqrt{DM_2} M_K\right) + 1$, with the comparison principle, we obtain (53) with

$$\mu^2 = e^{mD}\left(\int_0^D |g(x,0)|^2 \mathrm{d}x + M_K^2\right). \tag{57}$$

Note that because from (28) $g(D,0) = K(\hat{\beta}(0), \hat{\theta}(0))$, a possible choice for the initial condition of g is $g(x,0) = K(\hat{\beta}(0), \hat{\theta}(0))$ for all $x \in [0, D]$, such that the boundary condition (28) is compatible with the initial condition $g(x,0)$. In this case, (57) becomes $\mu^2 = e^{mD} M_K^2 (1+D)$. From relations (16)–(30) and (48) and using Young's and Cauchy–Schwartz's inequalities [29], one can show that bounds (49)–(52) hold with

$$M_u = 2\left(1 + \mu^2\right) \tag{58}$$

$$M_X = 5\left(De^{2M_A D}|M_2|\mu^2 + 1\right) \tag{59}$$

$$M_w = 2\left(1 + \mu^2\right)\left(1 + M_{\hat{Z}}\right) \tag{60}$$

$$M_{\hat{Z}} = 2\left(1 + D^2 e^{2M_A D}|M_2|\right). \tag{61}$$

□

Before we construct a Lyapunov functional for proving stability of the closed-loop system, we state the following lemma, which is concerned with an important property of the projector operator defined in (19).

Lemma 2
The following holds for (19)

$$-\int_0^D \tilde{\xi}(x)\mathrm{Proj}\left\{\tau, \hat{\xi}, \xi^{\mathrm{n}}, \rho\right\}(x)\mathrm{d}x \leqslant -\int_0^D \tilde{\xi}(x)\tau(x)\mathrm{d}x. \tag{62}$$

Proof
From (19), it follows that

$$-\int_0^D \tilde{\xi}(x)\mathrm{Proj}\left\{\tau, \hat{\xi}, \xi^{\mathrm{n}}, \rho\right\}(x)\mathrm{d}x = -\int_0^D \tilde{\xi}(x)\tau(x)\mathrm{d}x + \begin{cases} \left\langle \hat{\xi} - \xi^{\mathrm{n}}, \tilde{\xi}\right\rangle \frac{\langle \hat{\xi} - \xi^{\mathrm{n}}, \tau\rangle}{\|\hat{\xi} - \xi^{n}\|^2}, & \text{if } \|\hat{\xi} - \xi^{\mathrm{n}}\|^2 = \rho \text{ and } \left\langle \hat{\xi} - \xi^{\mathrm{n}}, \tau\right\rangle > 0 \\ 0, & \text{otherwise} \end{cases} \tag{63}$$

We prove now that the second term in (63) is always nonpositive. Assume that $\|\hat{\xi} - \xi^{\mathrm{n}}\|^2 = \rho$ and $\langle \hat{\xi} - \xi^{\mathrm{n}}, \tau \rangle > 0$. Then it is sufficient to show that

$$\int_0^D \tilde{\xi}(x) \left(\hat{\xi}(x) - \xi^{\mathrm{n}}(x)\right) \mathrm{d}x \leqslant 0. \tag{64}$$

It holds that

$$\int_0^D \tilde{\xi}(x) \left(\hat{\xi}(x) - \xi^{\mathrm{n}}(x)\right) \mathrm{d}x = \int_0^D (\xi(x) - \xi^{\mathrm{n}}(x)) \left(\hat{\xi}(x) - \xi^{\mathrm{n}}(x)\right) \mathrm{d}x - \int_0^D \left(\hat{\xi}(x) - \xi^{\mathrm{n}}(x)\right)^2 \mathrm{d}x. \tag{65}$$

Using the fact that $\|\hat{\xi} - \xi^{\mathrm{n}}\|^2 = \rho$ and the Cauchy–Schwartz inequality, we obtain

$$\int_0^D \tilde{\xi}(x) \left(\hat{\xi}(x) - \xi^{\mathrm{n}}(x)\right) \mathrm{d}x \leqslant \sqrt{\int_0^D (\xi(x) - \xi^{\mathrm{n}}(x))^2 \, \mathrm{d}x}\,\sqrt{\rho} - \rho. \tag{66}$$

With (11) we obtain (64). □

Remark 1

Instead of assuming in Assumption 1 (and respectively in the controllability condition of Assumption 2) that the $b_i(x)$ satisfy (11), one can assume that the $b_i(x)$ satisfy

$$R_{\mathrm{low},i}(x) \leqslant b_i(x) \leqslant R_{\mathrm{high},i}(x), \tag{67}$$

for some known, continuous functions $R_{\mathrm{low},i}(x)$ and $R_{\mathrm{high},i}(x)$. The new projection operator is

$$\mathrm{Proj}\{\tau, \hat{\xi}, R_{\mathrm{low}}, R_{\mathrm{high}}\}(x) = \begin{cases} 0, & \text{if } \hat{\xi}(x) = R_{\mathrm{low}}(x) \text{ and } \tau(x) < 0 \\ 0, & \text{if } \hat{\xi}(x) = R_{\mathrm{high}}(x) \text{ and } \tau(x) > 0 \\ \tau(x), & \text{otherwise} \end{cases} \tag{68}$$

Note that the projection set of the operator (68) is an infinite-dimensional hyper-rectangle, whereas the projection set of (19) is spherical. We show now that (68) satisfies

$$-\int_0^D \tilde{\xi}(x)\mathrm{Proj}\left\{\tau, \hat{\xi}, R_{\mathrm{low}}, R_{\mathrm{high}}\right\}(x)\mathrm{d}x \leqslant -\int_0^D \tilde{\xi}(x)\tau(x)\mathrm{d}x. \tag{69}$$

Using (68), we obtain

$$-\int_0^D \tilde{\xi}(x)\mathrm{Proj}\left\{\tau, \hat{\xi}, R_{\mathrm{low}}, R_{\mathrm{high}}\right\}(x)\mathrm{d}x = -\int_0^D \tilde{\xi}(x)\tau(x)\mathrm{d}x + \begin{cases} \int_0^D \tilde{\xi}(x)\tau(x)\mathrm{d}x, & \begin{array}{c} \text{if } \hat{\xi}(x) = R_{\mathrm{low}}(x) \text{ and } \tau(x) < 0 \\ \text{or} \\ \text{if } \hat{\xi}(x) = R_{\mathrm{high}}(x) \text{ and } \tau(x) > 0 \end{array} \\ 0, & \text{otherwise} \end{cases} \tag{70}$$

Assume first that $\hat{\xi}(x) = R_{\text{low}}(x)$ and $\tau(x) < 0$. As $\hat{\xi}(x) = R_{\text{low}}(x) \leqslant \xi(x)$, we obtain that

$$\tilde{\xi}(x)\tau(x) = \left(\xi(x) - \hat{\xi}(x)\right)\tau(x) \leqslant 0, \tag{71}$$

and hence, $\int_0^D \tilde{\xi}(x)\tau(x)\mathrm{d}x \leqslant 0$. The case $\hat{\xi}(x) = R_{\text{high}}(x)$ and $\tau(x) > 0$ can be proved analogously.

Lemma 3
Let γ_θ and γ_b be as in (31). Then for the Lyapunov function

$$V(t) = \log\left(1 + \Xi(t)\right) + \frac{1}{\gamma_b}\int_0^D \tilde{b}(x,t)^{\mathrm{T}}\tilde{b}(x,t)\mathrm{d}x + \frac{1}{\gamma_\theta}\tilde{\theta}(t)^{\mathrm{T}}\tilde{\theta}(t), \tag{72}$$

the following holds

$$V(t) \leqslant V(0). \tag{73}$$

Proof
Differentiating $V(t)$ with respect to time and using (2), (3) and (42), (45) we obtain

$$\begin{aligned}
\dot{V}(t) \leqslant{} & \frac{1}{1+\Xi(t)}\left(-\underline{\lambda}\left|\hat{Z}(t)\right|^2 + 2\hat{Z}(t)^{\mathrm{T}}P\left(\hat{\beta},\hat{\theta}\right)\int_0^D\int_0^x e^{-\hat{A}(x-y)}\sum_{i=1}^p B_i\dot{\hat{b}}_i(y,t)\mathrm{d}y\,u(x,t)\mathrm{d}x\right.\\
& + \left(\left|\dot{\hat{\theta}}(t)\right| + \int_0^D\left|e^{-\hat{A}(D-x)}\sum_{i=1}^p B_i\dot{\hat{b}}_i(x,t)\right|\mathrm{d}x\right)M_P'\left|\hat{Z}(t)\right|^2 - 2\hat{Z}(t)^{\mathrm{T}}P\left(\hat{\beta},\hat{\theta}\right)\\
& \times\sum_{i=1}^p\dot{\hat{\theta}}_i(t)A_i\int_0^D\int_0^x (x-y)e^{-\hat{A}(x-y)}\hat{B}(y,t)\mathrm{d}y\,u(x,t)\mathrm{d}x - w(0,t)^2 - \int_0^D w(x,t)^2\mathrm{d}x\\
& - 2\int_0^D(1+x)w(x,t)g(x,t)\mathrm{d}x\left(\int_0^D\int_0^x e^{-\hat{A}(x-y)}\sum_{i=1}^p B_i\dot{\hat{b}}_i(y,t)\mathrm{d}y\,u(x,t)\mathrm{d}x\right.\\
& \left. - \sum_{i=1}^p\dot{\hat{\theta}}_i(t)A_i\int_0^D\int_0^x(x-y)e^{-\hat{A}(x-y)}\hat{B}(y,t)\mathrm{d}y\,u(x,t)\mathrm{d}x\right) + 2\hat{Z}(t)^{\mathrm{T}}P\left(\hat{\beta},\hat{\theta}\right)\\
& \times\sum_{i=1}^p A_i\hat{Z}(t)\tilde{\theta}_i(t) - 2\hat{Z}(t)^{\mathrm{T}}P\left(\hat{\beta},\hat{\theta}\right)\sum_{i=1}^p A_i\int_0^D\int_0^x e^{-\hat{A}(x-y)}\hat{B}(y,t)\mathrm{d}y\,u(x,t)\mathrm{d}x\,\tilde{\theta}_i\\
& + 2\hat{Z}(t)^{\mathrm{T}}P\left(\hat{\beta},\hat{\theta}\right)\int_0^D\sum_{i=1}^p B_i\tilde{b}_i(x)u(x,t)\mathrm{d}x - 2\int_0^D(1+x)w(x,t)g(x,t)\mathrm{d}x\\
& \times\int_0^D\sum_{i=1}^p B_i\tilde{b}_i(x)u(x,t)\mathrm{d}x + 2\int_0^D(1+x)w(x,t)g(x,t)\mathrm{d}x\\
& \left.\times\sum_{i=1}^p A_i\left(\int_0^D\int_0^x e^{-\hat{A}(x-y)}\hat{B}(y,t)\mathrm{d}y\,u(x,t)\mathrm{d}x - \hat{Z}(t)\right)\tilde{\theta}_i\right)\\
& - \frac{2}{\gamma_b}\int_0^D\sum_{i=1}^p\tilde{b}_i(x,t)\dot{\hat{b}}_i(x,t)\mathrm{d}x - \frac{2}{\gamma_\theta}\sum_{i=1}^p\tilde{\theta}_i(t)\dot{\hat{\theta}}_i(t).
\end{aligned} \tag{74}$$

With (17)–(18), (47), and (62), we obtain

$$
\begin{aligned}
\dot V(t) \leqslant{} & \frac{1}{1+\Xi(t)}\left(-\underline{\lambda}\left|\hat Z(t)\right|^2+\left(\left|\dot{\hat\theta}(t)\right|+\int_0^D\left|e^{-\hat A(D-x)}\sum_{i=1}^p B_i\dot{\hat b}_i(x,t)\right|\mathrm{d}x\right)M_P'\left|\hat Z(t)\right|^2\right.\\
& +2\hat Z(t)^{\mathrm T}P\left(\hat\beta,\hat\theta\right)\int_0^D\int_0^x e^{-\hat A(x-y)}\sum_{i=1}^p B_i\dot{\hat b}_i(y,t)\mathrm{d}y\left(w(x,t)+g(x,t)\hat Z(t)\right)\mathrm{d}x\\
& -2\hat Z(t)^{\mathrm T}P\left(\hat\beta,\hat\theta\right)\sum_{i=1}^p\dot{\hat\theta}_i(t)A_i\int_0^D\int_0^x(x-y)e^{-\hat A(x-y)}\hat B(y,t)\mathrm{d}y\\
& \times\left(w(x,t)+g(x,t)\hat Z(t)\right)\mathrm{d}x-w(0,t)^2-\int_0^D w(x,t)^2\mathrm{d}x-2\int_0^D(1+x)w(x,t)\\
& \times g(x,t)\mathrm{d}x\left(\int_0^D\int_0^x e^{-\hat A(x-y)}\sum_{i=1}^p B_i\dot{\hat b}_i(y,t)\mathrm{d}y\left(w(x,t)+g(x,t)\hat Z(t)\right)\mathrm{d}x\right.\\
& \left.\left.-\sum_{i=1}^p\dot{\hat\theta}_i(t)A_i\int_0^D\int_0^x(x-y)e^{-\hat A(x-y)}\hat B(y,t)\mathrm{d}y\left(w(x,t)+g(x,t)\hat Z(t)\right)\mathrm{d}x\right)\right).
\end{aligned}
\tag{75}
$$

We now estimate $\int_0^D\left|\dot{\hat b}_i(x)\right|\mathrm{d}x$. Using (17), (19), and the Cauchy–Schwartz's inequality, we obtain that

$$
\int_0^D\left|\dot{\hat b}_i(x)\right|\mathrm{d}x\leqslant\gamma_b 2\sqrt{D}\sqrt{\int_0^D\tau_{b_i}(x)^2\mathrm{d}x}. \tag{76}
$$

We now estimate $\int_0^D\tau_{b_i}(x)^2\mathrm{d}x$. Using (23), (49), and (53) together with Young's and Cauchy–Schwartz's inequalities [29], we obtain

$$
\int_0^D\tau_{b_i}(x)^2\mathrm{d}x\leqslant 2M_u\left(M_P^2\left|B_i\right|^2+(1+D)^2\mu^2\right)\left(\frac{\left|\hat Z(t)\right|^2+\|w(t)\|^2}{1+\Xi(t)}\right)^2. \tag{77}
$$

With (76), we arrive at

$$
\int_0^D\left|\dot{\hat b}_i(x)\right|\mathrm{d}x\leqslant\gamma_b M_{\tau_{b_i}}\frac{\left|\hat Z(t)\right|^2+\|w(t)\|^2}{1+\Xi(t)}, \tag{78}
$$

where

$$
M_{\tau_{b_i}}=2\sqrt{2DM_u\left(M_P^2\left|B_i\right|^2+(1+D)^2\mu^2\right)}. \tag{79}
$$

With similar arguments, one can prove that

$$
\left|\dot{\hat\theta}_i(t)\right|\leqslant\gamma_\theta M_{\theta_i}\frac{\left|\hat Z(t)\right|^2+\|w(t)\|^2}{1+\Xi(t)}, \tag{80}
$$

where

$$
M_{\theta_i}=2|A_i|\left(\sqrt{1+D}\mu+M_P\right)\left(De^{M_A D}\sqrt{M_2}(1+\mu)+1\right). \tag{81}
$$

Using (78) and (80) together with Young's inequality [29], one can prove that there exists a positive constant M such that the following holds

$$\dot{V}(t) \leqslant \frac{1}{1+\Xi(t)}\left(-\underline{\lambda}\left|\hat{Z}(t)\right|^2 - w(0,t)^2 - \int_0^D w(x,t)^2 \mathrm{d}x \right.$$
$$\left. + \frac{(\gamma_\theta+\gamma_b)\,M}{1+\Xi(t)}\left(\int_0^D w(x,t)^2\mathrm{d}x + \left|\hat{Z}(t)\right|^2\right)^2\right), \tag{82}$$

where

$$M = M_P'\left(e^{M_A D}\sum_{i=1}^{p}|B_i|M_{\tau_{b_i}} + M_{\theta_i}\sum_{i=1}^{p}\right) + 2M_P D e^{M_A D}\sum_{i=1}^{p}|B_i|M_{\tau_{b_i}}\,(1+\mu) + 2M_P D$$
$$\times e^{M_A D}\sum_{i=1}^{p}|A_i|M_{\theta_i}\sqrt{M_2}\,(1+\mu) + 2(1+D)\mu e^{M_A D}\sum_{i=1}^{p}|B_i|M_{\tau_{b_i}}\,(D+1+\mu)$$
$$2D(1+D)\sqrt{M_2}\mu e^{M_A D}\sum_{i=1}^{p}|A_i|M_{\theta_i}\,(D+1+\mu)\,. \tag{83}$$

As from the definition of $\Xi(t)$ in (26), we have

$$1+\Xi(t) \geqslant \min\{\underline{\lambda},1\}\left(\left|\hat{Z}(t)\right|^2 + \int_0^D w(x,t)^2\mathrm{d}x\right), \tag{84}$$

we conclude that

$$\dot{V}(t) \leqslant \frac{1}{1+\Xi(t)}\left(-\underline{\lambda}\left|\hat{Z}(t)\right|^2 - w(0,t)^2 - \int_0^D w(x,t)^2 \mathrm{d}x \right.$$
$$\left. + (\gamma_\theta+\gamma_b)\frac{M}{\min\{\underline{\lambda},1\}}\left(\int_0^D w(x,t)^2\mathrm{d}x + \left|\hat{Z}(t)\right|^2\right)\right). \tag{85}$$

Choosing γ_θ and γ_b as in (31), the proof of the lemma is complete. □

Lemma 4
There exist constants $\underline{M}$ and $\overline{M}$ such that

$$\underline{M}\,\Xi(t) \leqslant \Pi(t) \leqslant \overline{M}\,\Xi(t), \tag{86}$$

where

$$\Pi(t) = |X(t)|^2 + \|u(t)\|^2. \tag{87}$$

Proof
Immediate, using (49)–(52) and the definition of $\Xi(t)$ in (26). □

We are now ready to derive the stability estimate of Theorem 1. Using (72), it follows that

$$\Xi(t) \leqslant \left(e^{V(t)}-1\right) \tag{88}$$

$$\left\|\tilde{b}(t)\right\|^2 + \left|\tilde{\theta}(t)\right|^2 \leqslant (\gamma_\theta+\gamma_b)\,V(t) \leqslant (\gamma_\theta+\gamma_b)\left(e^{V(t)}-1\right). \tag{89}$$

Consequently,

$$\Omega(t) \leqslant \left(\overline{M}+(\gamma_\theta+\gamma_b)\right)\left(e^{V(t)}-1\right). \tag{90}$$

Moreover, from (72), we take

$$V(0) \leqslant \max\left\{\overline{\lambda}, 1\right\}\left(|\hat{Z}(0)|^2 + \|w(0)\|^2\right) + \frac{1}{\gamma_b}\left\|\tilde{b}(0)\right\|^2 + \frac{1}{\gamma_\theta}\left|\tilde{\theta}(0)\right|^2. \tag{91}$$

Thus, by setting

$$R = \overline{M} + \gamma_\theta + \gamma_b \tag{92}$$

$$\rho = \max\left\{\overline{\lambda}, 1, \frac{1}{\gamma_\theta}, \frac{1}{\gamma_b}\right\}, \tag{93}$$

we obtain the stability result in Theorem 1.

We now turn our attention to proving the convergence of $X(t)$ and $U(t)$ to zero. We use here an alternative to Barbalat's lemma from Appendix A (which is proved in [30]). We first point out that from (73) it follows that $|\hat{Z}(t)|$, $\|w(t)\|$, $\left\|\hat{b}(t)\right\|$, and $\left|\hat{\theta}(t)\right|$ are uniformly bounded. From (15), it follows that $U(t)$ is uniformly bounded. From (4) and (49)–(50), we conclude that $\frac{\mathrm{d}X^2(t)}{\mathrm{d}t}$ is uniformly bounded. Finally, as from (85) it turns out that $|\hat{Z}(t)|$ and $\|w(t)\|$ are square integrable, using (50) and the alternative to Barbalat's lemma from Appendix A, we conclude that $\lim_{t\to\infty} X(t) = 0$. We now turn our attention to proving convergence of $U(t)$. With the help of (49) and by the square integrability of $|\hat{Z}(t)|$, we conclude using (15) that $U(t)$ is square integrable. It only remains to show that $\frac{\mathrm{d}U^2(t)}{\mathrm{d}t}$ is uniformly bounded. Hence, it is sufficient to show that $\dot{U}(t)$ is uniformly bounded. From (15) and with (42), as $\|\dot{\hat{b}}(t)\|$, $\|\tilde{b}(t)\|$ and $|\dot{\hat{\theta}}(t)|$, $|\tilde{\theta}(t)|$ are uniformly bounded, we conclude the uniform boundness of $\frac{\mathrm{d}U^2(t)}{\mathrm{d}t}$.

3. DISTURBANCE REJECTION FOR LTI PLANTS WITH KNOWN PARAMETERS

We consider the system

$$\dot{X}(t) = AX(t) + \int_0^D B(x)\left(u(x,t) + d\right)\mathrm{d}x \tag{94}$$

$$u_t(x,t) = u_x(x,t), \quad x \in [0, D] \tag{95}$$

$$u(D,t) = U(t), \tag{96}$$

where we assume that d is an unknown nonzero constant, and the matrices A and $B(x)$ for all $x \in [0, D]$ are known. We also assume that the state $X(t)$ and the infinite-dimensional actuator state $U(\theta)$, for all $\theta \in [t-D, t]$ are measured.

Assumption 4

The pair $\left(A, \int_0^D e^{-A(D-x)}B(x)\mathrm{d}x\right)$ is completely controllable, and there exist a vector K and matrices P, Q symmetric and positive definite such that

$$\left(A + \int_0^D e^{A(D-x)}B(x)\mathrm{d}xK\right)^{\mathrm{T}} P + P\left(A + \int_0^D e^{A(D-x)}B(x)\mathrm{d}xK\right) = -Q. \tag{97}$$

The controller for system (94)–(96) is

$$U(t) = K\hat{Z}(t) - \hat{d}(t) \tag{98}$$

$$\hat{Z}(t) = X(t) + \int_0^D \int_0^x e^{-A(x-y)} B(y)\mathrm{d}y \left(u(x,t) + \hat{d}(t)\right) \mathrm{d}x. \tag{99}$$

The update law for the disturbance is given by

$$\dot{\hat{d}}(t) = \gamma \left(\hat{Z}(t)^{\mathrm{T}} P - \alpha \int_0^D (1+x) w(x,t) K e^{A_{\mathrm{cl}}(x-D)} \mathrm{d}x\right) \Gamma_1, \tag{100}$$

where

$$w(x,t) = u(x,t) - K e^{A_{\mathrm{cl}}(x-D)} \hat{Z}(t) + \hat{d}(t) \tag{101}$$

$$A_{\mathrm{cl}} = A + \int_0^D e^{A(D-x)} B(x)\mathrm{d}x K \tag{102}$$

$$\Gamma_1 = \int_0^D B(x)\mathrm{d}x \tag{103}$$

$$\alpha = \lambda_{\min}(Q). \tag{104}$$

Theorem 2
Consider the closed-loop systems consisting of the plant (94)–(96) together with the adaptive controller (98)–(103). Let Assumption 4 be satisfied and choose γ such that

$$\gamma \leqslant \frac{\lambda_{\min}(Q)}{8|\Gamma_1|\left(|P| + \alpha(1+D)|K|e^{|A_{\mathrm{cl}}|D}\right)\left(|P\Gamma_2| + \alpha(1+D)\left(1 + |K|e^{|A_{\mathrm{cl}}|D}|\Gamma_2|\right)\right)}. \tag{105}$$

Then there exists a positive constant R such that

$$|X(t)|^2 + \|u(t)\|^2 + \tilde{d}(t)^2 \leqslant R\left(|X(0)|^2 + \|u(0)\|^2 + \tilde{d}(0)^2 + d^2\right), \tag{106}$$

where

$$\tilde{d}(t) = d - \hat{d}(t). \tag{107}$$

Furthermore,

$$\lim_{t\to\infty} X(t) = 0 \tag{108}$$

$$\lim_{t\to\infty} U(t) = -d. \tag{109}$$

We employ here similar arguments as in Section 2. Differentiating (99) with respect to time and using (94)–(96) together with (98), we obtain that

$$\dot{\hat{Z}}(t) = A_{\mathrm{cl}}\hat{Z}(t) + \Gamma_1\tilde{d}(t) + \Gamma_2\dot{\hat{d}}(t), \tag{110}$$

where

$$\Gamma_2 = \int_0^D \int_0^x e^{-A(x-y)} B(y)\mathrm{d}y\mathrm{d}x, \tag{111}$$

and we used also the fact that $-A\Gamma_2 = -\Gamma_1 + \int_0^D e^{-A(D-x)} B(x)\mathrm{d}x$ (which follows from the fact that $F'(x) = B(x) - AF(x)$, with $F(x) = \int_0^x e^{-A(x-y)} B(y)\mathrm{d}y$). Differentiating (101) with respect to time and with respect to the spatial variable x and by using (95)–(96), (98) and (110), we obtain

$$w_t(x,t) = w_x(x,t) - K e^{A_{\mathrm{cl}}(x-D)}\Gamma_1\tilde{d}(t) + \left(1 - K e^{A_{\mathrm{cl}}(x-D)}\Gamma_2\right)\dot{\hat{d}}(t) \tag{112}$$

$$w(D,t) = 0. \tag{113}$$

Analogously with Lemma 1, we have the following lemma.

Lemma 5
There exist constants F_u, F_w, F_X, and $F_{\hat{Z}}$ such that

$$\|u(t)\|^2 \leqslant F_u \left(\|w(t)\|^2 + |\hat{Z}(t)|^2 + |\hat{d}(t)|^2\right) \tag{114}$$

$$|X(t)|^2 \leqslant F_X \left(|\hat{Z}(t)|^2 + \|w(t)\|^2\right) \tag{115}$$

$$\|w(t)\|^2 \leqslant F_w \left(\|u(t)\|^2 + |X(t)|^2 + |\hat{d}(t)|^2\right) \tag{116}$$

$$|\hat{Z}(t)|^2 \leqslant F_{\hat{Z}} \left(|X(t)|^2 + \|u(t)\|^2 + |\hat{d}(t)|^2\right). \tag{117}$$

Proof
Using relation (101) together with Young's and Cauchy–Schwartz's inequalities [29], we obtain bound (114) with

$$F_u = 3(1+D)\left(2 + |K|^2 e^{2|A_{\mathrm{cl}}|D}\right). \tag{118}$$

Substituting $u(x,t) + \hat{d}(t) = w(x,t) + Ke^{A_{\mathrm{cl}}(x-D)}\hat{Z}(t)$ into (99) and solving for $X(t)$, we obtain bound (115) with

$$F_X = 2\left(\left(1 + |\Gamma_2||K|e^{|A_{\mathrm{cl}}|D}\right)^2 + D^2 e^{2|A|D}\left(\int_0^D |B(x)|\mathrm{d}x\right)^2\right). \tag{119}$$

From (99) and with Young's and Cauchy–Schwartz's inequalities [29], we obtain bound (117) with

$$F_{\hat{Z}} = 2\left((1+|\Gamma_2|)^2 + D^2 e^{2|A|D}\left(\int_0^D |B(x)|\mathrm{d}x\right)^2\right). \tag{120}$$

From (101) and (117), we obtain bound (116) with

$$F_w = 3\left(1 + D + |K|^2 e^{2|A_{\mathrm{cl}}|D} F_{\hat{Z}}\right). \tag{121}$$

□

Lemma 6
Let γ and α be as in (105). Then for the Lyapunov function

$$V(t) = \hat{Z}(t)^{\mathrm{T}} P \hat{Z}(t) + \alpha\int_0^D (1+x)w(x,t)^2\mathrm{d}x + \frac{1}{\gamma}\tilde{d}(t)^2, \tag{122}$$

the following holds

$$V(t) \leqslant V(0). \tag{123}$$

Proof
Differentiating $V(t)$ with respect to time and using (110), (112), and (113) and integrating by parts the w integral, we obtain

$$\begin{aligned}\dot{V}(t) \leqslant & -\lambda_{\min}(Q)|\hat{Z}(t)|^2 - \alpha w(0,t)^2 - \alpha \int_0^D w(x,t)^2 \mathrm{d}x - \frac{2}{\gamma}\dot{\hat{d}}\tilde{d} \\ & + 2\left(\hat{Z}(t)^{\mathrm{T}} P - \alpha \int_0^D (1+x) w(x,t) K e^{A_{\mathrm{cl}}(x-D)} \mathrm{d}x\right)\Gamma_1 \tilde{d} \\ & + 2\left(\hat{Z}(t)^{\mathrm{T}} P \Gamma_2 + \alpha \int_0^D (1+x) w(x,t)\left(1 - K e^{A_{\mathrm{cl}}(x-D)}\Gamma_2\right)\mathrm{d}x\right)\dot{\hat{d}}.\end{aligned} \tag{124}$$

With (100) and using the fact that $|\dot{\hat{d}}| \leqslant \gamma\delta_1\left(|\hat{Z}| + \int_0^D |w(x,t)|\mathrm{d}x\right)$ where $\delta_1 = |P\Gamma_1| + \alpha(1+D)|Ke^{|A_{\mathrm{cl}}|D}\Gamma_1|$, we obtain

$$\begin{aligned}\dot{V}(t) \leqslant & -\lambda_{\min}(Q)|\hat{Z}(t)|^2 - \alpha \int_0^D w(x,t)^2 \mathrm{d}x - \alpha w(0,t)^2 \\ & + \gamma\delta_1\delta_2\left(|\hat{Z}(t)|^2 + \int_0^D w(x,t)^2 \mathrm{d}x\right),\end{aligned} \tag{125}$$

where we used the fact that $(p+r)^2 \leqslant 2(p^2+r^2)$ and $\delta_2 = 4|P\Gamma_2| + 4\alpha(1+D)\left(1 + |Ke^{|A_{\mathrm{cl}}|D}\Gamma_2|\right)$. Choosing γ and α as in (105)–(104), we have that

$$\dot{V}(t) \leqslant -\frac{\lambda_{\min}(Q)}{2}|\hat{Z}(t)|^2 - \frac{\alpha}{2}\int_0^D w(x,t)^2 \mathrm{d}x - \alpha w(0,t)^2. \tag{126}$$

□

We are now ready to derive the stability estimate of Theorem 2. Using relations (118), (119) and (122), (123), it follows that

$$\begin{aligned}|X(t)|^2 + \|u(t)\|^2 + \tilde{d}(t)^2 & \leqslant 2\left(F_X + F_u\right)\left(\|w(t)\|^2 + |\hat{Z}(t)|^2 + |\tilde{d}(t)|^2 + d^2\right) \\ & \leqslant 2\left(F_X + F_u\right)\left(\frac{1}{\min\left\{\lambda_{\min}(P), \alpha, \frac{1}{\gamma}\right\}} + 1\right)\left(V(0) + d^2\right).\end{aligned} \tag{127}$$

Moreover, using (122) and the bounds (120) and (121), we obtain

$$|X(t)|^2 + \|u(t)\|^2 + \tilde{d}(t)^2 \leqslant 2M_1\left(\|u(0)\|^2 + |X(0)|^2 + |\tilde{d}(0)|^2 + d^2\right) \tag{128}$$

where $M_1 = (F_X + F_u)\left(\lambda_{\max}(P) + \alpha(1+D) + \frac{1}{\gamma} + 1\right)\left(F_{\hat{Z}} + F_w + 1\right)\left(\frac{1}{\min\left\{\lambda_{\min}(P), \alpha, \frac{1}{\gamma}\right\}} + 1\right)$. Thus, by setting $R = 2M_1$, we obtain the stability result in Theorem 1.

We now turn our attention to proving the convergence of $X(t)$ and $U(t)$. We first point out that from (123), it follows that $|\hat{Z}(t)|$, $\|w(t)\|$ and $\left|\hat{d}(t)\right|$ are uniformly bounded. From (98), we obtain that $|U(t)|$ is uniformly bounded for all $t \geqslant 0$, and hence, $|u(0,t)|$ is uniformly bounded for all $t \geqslant D$. From (100), we obtain the uniform boundness of $|\dot{\hat{d}}(t)|$. Using the fact that $\frac{\mathrm{d}\int_0^D w(x,t)^2 \mathrm{d}x}{\mathrm{d}t} = 2\int_0^D w(x,t) w_t(x,t)\mathrm{d}x$, integration by parts and relation (112), we conclude that $\frac{\mathrm{d}\|w(t)\|^2}{\mathrm{d}t}$ is uniformly bounded if $|w(0,t)|$ is uniformly bounded. This fact follows from (101) and the uniform boundness of $|u(0,t)|$ for all $t \geqslant D$. As from (126) it turns out that $|\hat{Z}(t)|$ and $\|w(t)\|$

are square integrable and using the uniform boundness of $\frac{\mathrm{d}|\hat{Z}(t)|^2}{\mathrm{d}t}$ which follows from (110), using an alternative to Barbalat's lemma in Appendix A, we conclude that $\lim_{t\to\infty} |\hat{Z}(t)| = 0$ and that $\lim_{t\to\infty} \|w(t)\| = 0$. Using (115), we obtain the regulation of $|X(t)|$ to zero. We prove now the convergence of U. As $\int_0^\infty \dot{\hat{Z}}(t)\mathrm{d}t = \hat{Z}(\infty) - \hat{Z}(0)$ exists and is bounded and as $\ddot{\hat{Z}}(t)$ is uniformly bounded from (110), (100), and (112), we obtain that $\dot{\hat{Z}}(t)$ is uniformly continuous, and hence, with Barbalat's lemma, we obtain that $\lim_{t\to\infty} \dot{\hat{Z}}(t) = 0$. From (100), we obtain that $\lim_{t\to\infty} \dot{\hat{d}}(t) = 0$, and hence, using (110) and the fact that not all components of Γ_1 are zero (because in this case from (94) one can observe that d has no influence on the plant, and one can simply choose $\hat{d} = 0$), we obtain that $\lim_{t\to\infty} \tilde{d}(t) = 0$. Therefore, from (98), we obtain that $\lim_{t\to\infty} U(t) = -d$, which completes the proof.

4. SIMULATIONS

The method that is used to discretize (130) and (133) in space is the finite-difference method. The resulting finite-dimensional ODEs are solved using Euler's method. The integro-differential equations (129) and (132) are solved using Euler's method, where the integrals are computed using the left-endpoint rule for numerical integration.

4.1. Adaptive control

We consider the following scalar plant

$$\dot{X}(t) = \theta X(t) + \int_0^1 (1.5 + \sin(5x))\, u(x,t)\mathrm{d}x \tag{129}$$

$$u_t(x,t) = u_x(x,t) \tag{130}$$

$$u(1,t) = U(t), \tag{131}$$

where $A = \theta = 1$ and $B_0(x) = 0$, $B_1 = 1$, $b_1(x) = 1.5 + \sin(5x)$. We choose the initial condition of the system as $X(0) = 1$ and $U(\theta) = 0$ for all $-D \leqslant \theta \leqslant 0$. We choose the parameters for the controller as $K = \frac{A_{\mathrm{cl}} - \hat{\theta}(t)}{\int_0^D e^{-\hat{\theta}(t)(D-x)}\hat{b}_1(x,t)\mathrm{d}x}$, $A_{\mathrm{cl}} = -0.7$, $\gamma_b = 0.003$, $\gamma_\theta = 0.001$ and $\rho_1 = 2$, $b_i^{\mathrm{n}}(x) = 1$ for all $x \in [0,1]$, $\underline{\theta} = 0.4\theta$, $\overline{\theta} = 2\theta$. We choose $\hat{\theta}(0) = \underline{\theta}$ and $\hat{b}_1(x,0) = \sqrt{\rho_1} + 1$ for all $x \in [0,1]$. Note that with this choice of initial estimates for b_1 and θ, A_{cl} is unstable ($A_{\mathrm{cl}} = 0.2698$) when $\gamma_b = \gamma_\theta = 0$. In Figure 1, we show the response of the system and in Figure 2 the estimates of θ and $b_1(x)$. Finally, in Figure 3, we show the final profile of the estimate $\hat{b}_1(x,t)$ versus the real $b_1(x)$, as well as the estimation of the norm $\|b_1(t) - b_1^{\mathrm{n}}\|^2$.

4.2. Disturbance rejection

We consider the following scalar plant

$$\dot{X}(t) = \theta X(t) + b_1 \int_0^1 (u(x,t) + d)\, \mathrm{d}x \tag{132}$$

$$u_t(x,t) = u_x(x,t) \tag{133}$$

$$u(1,t) = U(t), \tag{134}$$

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where $\theta = 1$ and $b_1 = 0.5$. We choose the initial condition of the system as $X(0) = 1$ and $U(\theta) = 0$ for all $-D \leqslant \theta \leqslant 0$. The parameters of the control law (98)–(103) are chosen as $\gamma = 0.001$, $A_{\rm cl} = -2$, and $K = \theta \frac{A_{\rm cl}-\theta}{b_1(1-e^{-\theta})}$. We assume initially that a constant disturbance of magnitude $d = 0.5$ perturbs the closed-loop system at time $t = 10$ s. In Figure 4, we show the response of the system and the control effort, and in Figure 5 the estimation of the unknown disturbance. The control law compensates the disturbance and brings the state of the plant to zero.

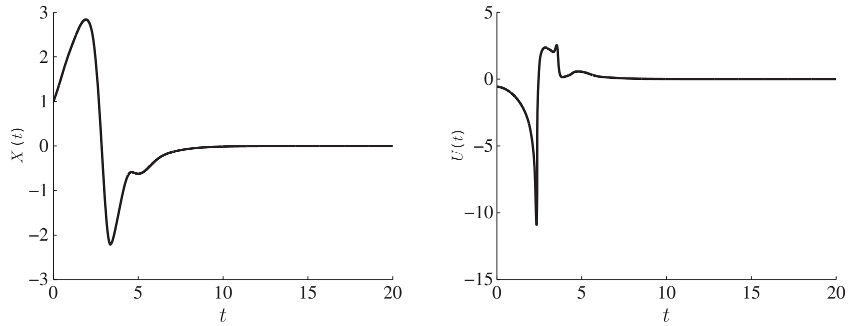

Figure 1. Response of the plant (129)–(131) with the adaptive control algorithm (15)–(30), for initial conditions $X(0) = 1$, $U(\theta) = 0$ for all $-D \leqslant \theta \leqslant 0$ and $\hat{b}_1(x, 0) = \sqrt{\rho_1} + 1$ for all $x \in [0, 1]$, $\hat{\theta}(0) = \underline{\theta}$.

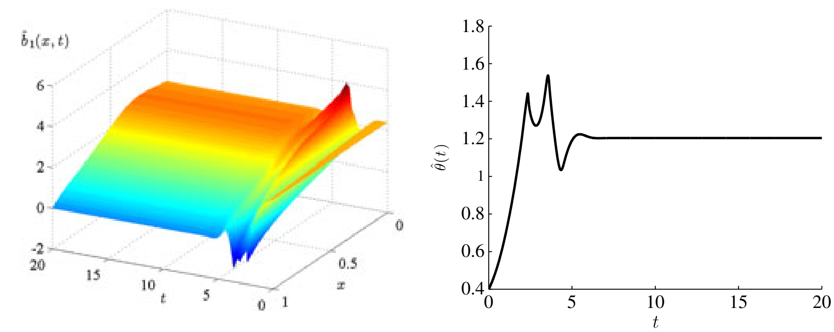

Figure 2. The estimation of the parameters b_1 (left) and θ (right), for initial conditions $X(0) = 1$, $U(\theta) = 0$ for all $-D \leqslant \theta \leqslant 0$ and $\hat{b}_1(x, 0) = \sqrt{\rho_1} + 1$ for all $x \in [0, 1]$, $\hat{\theta}(0) = \underline{\theta}$.

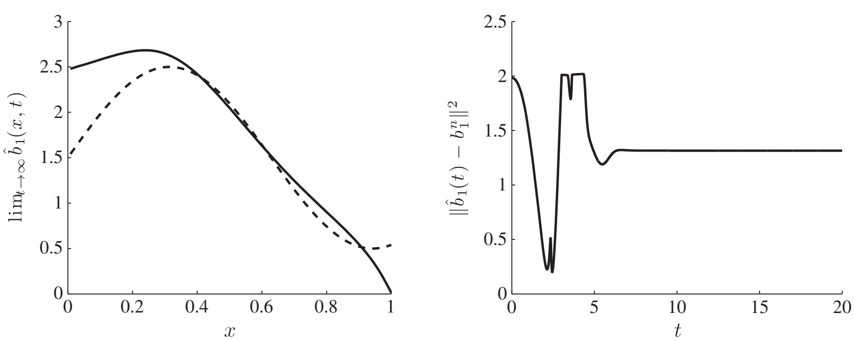

Figure 3. Left: The final profile of the estimate $\hat{b}_1(x, t)$ (solid) versus true $b_1(x)$ (dashed). Right: The estimate $\int_0^D \left(\hat{b}_1(x, t) - b_1^{\rm n}(x)\right)^2 {\rm d}x$.

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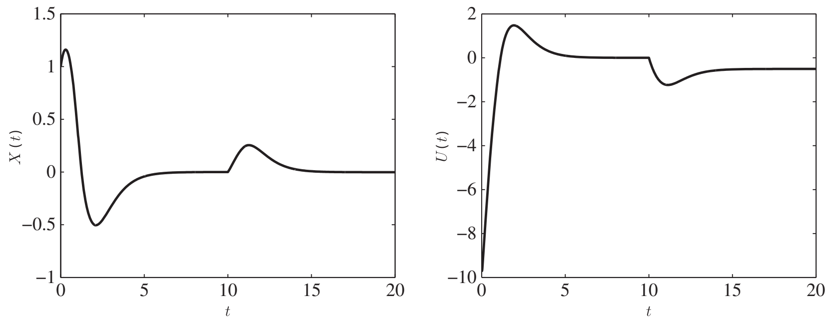

Figure 4. Response of the plant (132)–(134) with the disturbance rejection control algorithm (98)–(103), for initial conditions $X(0) = 1$ and $U(\theta) = 0$ for all $-D \leqslant \theta \leqslant 0$ under a step disturbance $d = 0.5$ applied at $t = 10$ s.

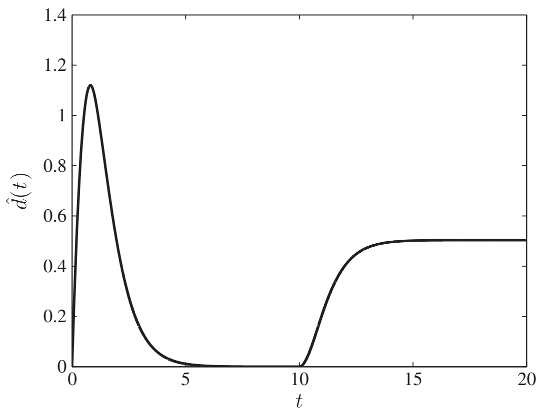

Figure 5. Estimation of the disturbance d, for initial conditions $X(0) = 1$ and $U(\theta) = 0$ for all $-D \leqslant \theta \leqslant 0$ under a step disturbance $d = 0.5$ applied at $t = 10$ s.

5. CONCLUSIONS

We present an approach for stabilizing linear systems with distributed input delay and unknown plant parameters. We employ update laws on the basis of the construction of a Lyapunov function with normalization. For linear systems with distributed input delay and an unknown constant disturbance in the input, we design an adaptive controller that compensates the disturbance and achieves regulation, when the plant parameters are known.

APPENDIX A: AN ALTERNATIVE TO BARBALAT'S LEMMA

Let the function $f(t) : \mathbb{R}_+ \to \mathbb{R}_+$ be differentiable and such that

$$f'(t) \leqslant B_f, \quad \text{for all } t \geqslant 0 \tag{A.1}$$

$$\int_0^\infty f(t)\mathrm{d}t < \infty, \tag{A.2}$$

for some positive constant B_f. Then

$$\lim_{t\to\infty} f(t) = 0. \tag{A.3}$$

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