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Nonlinear control under delays that depend on delayed states

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ABSTRACT

We recently introduced a design methodology for the stabilization of nonlinear systems with input and state delays that depend on the *current* state of the plant. In the present paper we consider nonlinear systems with delays that depend on the *delayed* state of the plant, that is, the delay is defined implicitly as a nonlinear function of the state at a past time which depends on the delay itself. Since the prediction horizon and the delay depend on the state of the plant, the key design challenges are how to compute the predictor state and the delay (since the delay needs to be available in order to compute the predictor). We resolve these challenges and we establish closed-loop stability with the aid of a strict Lyapunov functional that we construct. We also design a predictor feedback law for systems with state delays that depend on delayed states. We present an example of a strict-feedforward nonlinear system with input delay.

1. Introduction

The time that is required for a relativistic particle to feel the electromagnetic force of another particle depends on the position of the particles at a past time instant [29]. The round trip time of a signal in a network is a past value of the queuing delay at a time that depends on past values of the queuing delay itself [7,8]. The common attribute of these two time periods is that they depend on past values of the state of the system itself. In other words, these two time periods are examples of state-dependent delays that depend on delayed states.

Input delay compensation for nonlinear systems is achieved using predictor feedback [13–15,20]. Alternative control designs for nonlinear systems with constant delays can be found in [10,11,21–26]. Predictor feedback has been also successful in compensation of time-varying delays [2,19]. An alternative design for nonlinear systems with time-varying delays is the one in [12]. The predictor feedback design for nonlinear systems with state-dependent delays was provided recently [3]. Results dealing with the robustness of predictor feedback to time-varying [16,4], state-dependent [4] or input-dependent delays [5,6] also exist.

In [3] we developed a systematic methodology for the compensation of input delays that depend on the current state of the plant. In this paper we consider a different problem in which the delay depends on past values of the state, at a time that depends on the delay itself. Since the delay is needed for computing the predictor state the first design challenge that we resolve is computing the delay through this implicit relation. Since the prediction horizon, over which we design the predictor, depends on the state of the system, the second design challenge that we resolve is computing the predictor state. We then present the predictor feedback law for general nonlinear systems (Section 2). The real complexity of the problem is in the stability analysis of the closed-loop system. Due to an inherent limitation that the delay rate is larger than -1 (which ensures that the delay function is uniquely defined, and hence, that the dynamical system which describes the dynamics of the plant is uniquely defined) and since the delay depends on the state, only local results are possible. We prove local asymptotic stability of the closed-loop system with the aid of a Lyapunov functional that we construct by introducing a backstepping transformation of the actuator state (Section 3). We present a numerical example of a second-order strict-feedforward system with a state-dependent input delay (Section 4). We also consider nonlinear systems in the strict-feedback form with a delay on the virtual input that depends on past values of the state of the system at which the control input enters. For this class of systems we design the predictor feedback law and prove asymptotic stability of the closed-loop system (Section 5).

Notation: We use the common definition of class $\mathcal{K}$, $\mathcal{K}_\infty$ and $\mathcal{KL}$ functions from [17]. For an n-vector, the norm $|\cdot|$ denotes the usual Euclidean norm. We say that a function $\rho:\mathbb{R}_+\times(0,1)\mapsto\mathbb{R}_+$ belongs to class $\mathcal{KC}$ if it is of class $\mathcal{K}$ with respect to its first argument for each value of its second argument and continuous with respect to its second argument. It belongs to class $\mathcal{KC}_\infty$ if it is in $\mathcal{KC}$ and also in $\mathcal{K}_\infty$ with respect to its first argument.

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2. Predictor feedback under input delays

2.1. Problem formulation

We consider the following plant:

$$\dot{X}(t) = f(X(t), U(\phi(t))), \tag{1}$$

where $t \geq \phi(0)$, $U \in \mathbb{R}$, $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is locally Lipschitz with $f(0,0) = 0$ and ϕ satisfies

$$t = \phi(t) + D(X(\phi(t))). \tag{2}$$

We impose the following assumptions on the delay function D and the plant (1).

Assumption 1. $D \in C^1(\mathbb{R}^n; \mathbb{R}_+)$ and ∇D is locally Lipschitz.[1]

Assumption 2. The plant $\dot{X} = f(X, \omega)$ is strongly forward complete, that is, there exist a smooth positive definite function R and class $\mathcal{K}_\infty$ functions α_1, α_2 and α_3 such that

$$\alpha_1(|X|) \leq R(X) \leq \alpha_2(|X|) \tag{3}$$

$$\frac{\partial R(X)}{\partial X} f(X, \omega) \leq R(X) + \alpha_3(|\omega|), \tag{4}$$

for all $X \in \mathbb{R}^n$ and for all $\omega \in \mathbb{R}$.

This property differs from the standard forward completeness [1] in that we assume that $f(0,0) = 0$ and hence, $R(\cdot)$ is positive definite. Assumption 2 guarantees that system (1) does not exhibit finite escape time, that is, for every initial condition and every locally bounded input signal the corresponding solution is defined for all times.

Assumption 3. The plant $\dot{X} = f(X, \kappa(X) + \omega)$ is input-to-state stable with respect to ω and the function κ is locally Lipschitz with $\kappa(0) = 0$.

2.2. Predictor feedback design

We refer to the quantity $t - \phi(t) = D(X(\phi(t)))$ as "delay". This is the time interval that indicates how long ago the control signal that currently affects the plant was actually applied. Consequently, the delay D depends on the value of the state at the time the control was applied. The goal of our predictor-based design is to completely compensate this input delay, that is, after the control signal reaches the plant, i.e., when $\phi(t) \geq 0$ (which happens for first time at $t^* = D(X(0))$), to make the closed-loop system to behave as if there were no delay at all. To achieve this we first have to appropriately define the predictor of the state X, that is, the signal that satisfies

$$P(\phi(t)) = X(t) \quad \text{for all } t \geq 0. \tag{5}$$

Assume for the moment that $\phi'(t) > 0$, for all $t \geq 0$ (we show later on that under a sufficient condition, which incorporates the delay function D and the initial conditions and solutions of the system, that this is true), which in particular implies that ϕ is invertible. Denoting

$$\sigma(\theta) = \phi^{-1}(\theta) \quad \text{for all } \phi(t) \leq \theta \leq t, \tag{6}$$

the predictor state P is $P(\theta) = X(\sigma(\theta))$ for all $\phi(t) \leq \theta \leq t$. With the help of (2) we have

$$\sigma(\theta) = \theta + D(X(\theta)) \quad \text{for all } \phi(t) \leq \theta \leq t. \tag{7}$$

Therefore, the predictor of X, $P(t) = X(\sigma(t))$ is

$$P(t) = X(t + D(X(t))) \quad \text{for all } t \geq 0. \tag{8}$$

Having defined the predictor of X we now need to compute this signal. This is not an easy task since P cannot be directly computed from relation (8), because P depends on the future values of X which are not available. In addition to that, the quantity $\sigma(t) - t = D(X(t))$, which from now on we refer to as the prediction horizon (this is the time interval which indicates after how long an input signal that is currently applied affects the plant), depends on the state $X(t)$. Note here that the delay time $D(X(\phi(t)))$ is not equal to the prediction horizon $D(X(t))$.

We are now ready to compute P. Since the predictor state $P(\theta)$, $\phi(t) \leq \theta \leq t$ satisfies $P(\theta) = X(\sigma(\theta))$ we perform a change of variables $t = \sigma(\theta)$ in (1) and using definition $\sigma(\theta) = \theta + D(X(\theta))$ we get that

$$\frac{dP(\theta)}{d\theta} = (1 + \nabla D(X(\theta)) f(X(\theta), U(\phi(\theta)))) f(P(\theta), U(\theta)). \tag{9}$$

Integrating this relation from $\phi(t)$ to t and using the fact that $P(\phi(t)) = X(t)$ we get for all $\phi(t) \leq \theta \leq t$

$$P(\theta) = X(t) + \int_{\phi(t)}^{\theta} (1 + \nabla D(X(s)) f(X(s), U(\phi(s)))) f(P(s), U(s))\, ds. \tag{10}$$

From relation (10) one can observe that for computing $P(t)$, besides having available $X(t)$ and the history of the signals $X(\sigma)$, $U(\sigma)$, $U(\phi(\sigma))$, $P(\sigma)$ on the interval $[\phi(t), t)$ one needs to know the function $\phi(t)$. We compute next $\phi(t)$, which is defined implicitly through relation (2). We proceed analogously with the derivation of relation (10). We define the change of variables $t = \theta$, for all $t \leq \theta \leq t + D(X(t))$ and differentiate (2) to get

$$1 = \phi'(\theta) + \nabla D(X(\phi(\theta))) f(X(\phi(\theta)), U(\phi(\phi(\theta)))) \phi'(\theta). \tag{11}$$

Solving for ϕ' and integrating backward the resulting relation starting at the known value $\phi(t + D(X(t))) = t$, which follows from (6) and (7), we arrive for all $t \leq \theta \leq t + D(X(t))$ at

$$\phi(\theta) = t - \int_{\theta}^{t + D(X(t))} \frac{ds}{1 + \nabla D(X(\phi(s))) f(X(\phi(s)), U(\phi(\phi(s))))}. \tag{12}$$

The predictor-based controller is now derived based on the delay-free design (see Assumption 3) as

$$U(t) = \kappa(P(t)). \tag{13}$$

One can observe from (12) that the denominator has to be nonzero. What is more, $\phi(\theta)$ has to be invertible for all $t \leq \theta \leq t + D(X(t))$. A sufficient condition, on the initial conditions and the solutions of the system, for these two facts to hold simultaneously is

$$\mathcal{F}_c : c + \nabla D(X(\phi(s))) f(X(\phi(s)), U(\phi(\phi(s)))) > 0 \quad \text{for all } s \geq 0, \tag{14}$$

for some $c \in (0, 1]$. We refer to $\mathcal{F}_1$ as the *feasibility condition* of the controller (10), (12), (13). Although the actual feasibility region of the controller is given by the initial conditions and solutions of the system satisfying the feasibility condition $\mathcal{F}_1$ in (14), it turns out that the stability analysis is simplified if one imposes a more restrictive condition (which guarantees the satisfaction of condition (14)). In the subsequent development we impose the following condition on the initial conditions and solutions of the system

$$\mathcal{F}_c^* : |\nabla D(X(\phi(s))) f(X(\phi(s)), U(\phi(\phi(s))))| < c \quad \text{for all } s \geq 0, \tag{15}$$

for some $0 < c < 1$. It is evident that if $\mathcal{F}_c^*$ is satisfied then $\mathcal{F}_c$ is also satisfied.

Note that the requirement $\dot{D} = \nabla D f > -1$ is an inherent limitation of the plant and not a restriction of the control design. This condition guarantees that $0 < \phi' < \infty$, that is, it guarantees that ϕ is a single-valued function, which in turn ensures that the dynamical system which describes the dynamics of the plant is uniquely defined.

[1] To ensure uniqueness of solutions.

2.3. Implementation

In an actual implementation of the predictor-based control law (13), at each time step one has to compute the predictor P using relation (10) for $\theta = t$. The integral in (10) is computed from the history of $X(s)$, for all $s \in [\phi(t), t)$, and of $U(s)$, for all $s \in [\phi(\phi(t)), t)$, using a method of numerical integration, with a total number of $N_P(t) = D(X(\phi(t)))/h$ points, where h is the discretization step. However, for computing P one has to first compute ϕ. This computation is performed by calculating the integral in (12) from the history of $X(s)$, for all $s \in (\phi(t), t]$, and of $U(s)$, for all $s \in (\phi(\phi(t)), \phi(t)]$,[2] using a method of numerical integration with $N_\phi(t) = D(X(t))/h$ points. Alternatively, one could compute ϕ by numerically solving relation (2) with respect to ϕ.

3. Stability analysis under input delays

Theorem 1. *Consider the closed-loop system consisting of the plant (1) and the control law (10), (12), (13). Under Assumptions 1–3 there exist a class $\mathcal{K}$ function ψ_{RoA}, a class $\mathcal{KC}_\infty$ function ρ and a class $\mathcal{KL}$ function β such that for all initial conditions for which $X(\cdot)$, $U(\cdot)$, $U(\phi(\cdot))$ are locally Lipschitz on the interval $[\phi(0), 0)$, and which satisfy*

$$\Omega(0) < \psi_{\mathrm{RoA}}(c), \tag{16}$$

for some $0 < c < 1$, where

$$\Omega(t) = |X(t)| + \sup_{\phi(t) \le \theta \le t} |X(\theta)| + \sup_{\phi(\phi(t)) \le \sigma \le t} |U(\sigma)|, \tag{17}$$

there exists a unique solution to the closed-loop system with X Lipschitz on $[0, \infty)$, U Lipschitz on $(0, \infty)$, and

$$\Omega(t) \le \beta(\rho(\Omega(0), c), t), \tag{18}$$

for all $t \ge 0$. Furthermore, there exists a class $\mathcal{K}$ function δ^* such that, for all $t \ge 0$,

$$\sup_{\phi(t) \le \theta \le t} D(X(\theta)) \le D(0) + \delta^*(c) \tag{19}$$

$$\sup_{\phi(t) \le \theta \le t} |\dot{D}(X(\theta))| \le c. \tag{20}$$

The proof of Theorem 1 is based on a series of technical lemmas which are presented next.

Lemma 1. *The infinite-dimensional backstepping transformation of the actuator state defined for all $\phi(\phi(t)) \le \theta \le t$ by*

$$W(\theta) = U(\theta) - \kappa(P(\theta)), \tag{21}$$

together with the control law (13) transform the plant (1) to the "target system" given by

$$\dot{X}(t) = f(X(t), \kappa(X(t)) + W(\phi(t))) \quad \text{for all } t \ge \phi(0) \tag{22}$$

$$W(t) = 0 \quad \text{for all } t \ge 0. \tag{23}$$

Proof. Using (13) we get (23). Noting that for all $t \ge 0$, $P(\phi(t)) = X(t)$ and defining $P(\phi(t)) = X(t)$ also for $\phi(0) \le t \le 0$, from (21) we get $W(\phi(t)) = U(\phi(t)) - \kappa(X(t))$, for all $t \ge \phi(0)$, and hence, from (1) we get (22). □

Lemma 2. *The inverse transformation of (21) is given for all $\phi(\phi(t)) \le \theta \le t$ by*

$$U(\theta) = W(\theta) + \kappa(\Pi(\theta)), \tag{24}$$

where for all $\phi(t) \le \theta \le t$

$$\Pi(\theta) = X(t) + \int_{\phi(t)}^{\theta} (1 + \nabla D(X(\sigma)) f(X(\sigma), \kappa(X(\sigma)) + W(\phi(\sigma)))) \times f(\Pi(\sigma), \kappa(\Pi(\sigma)) + W(\sigma))\, d\sigma. \tag{25}$$

Proof. By direct verification we get that $\Pi(t) = P(t)$[3] for all $t \ge \phi(0)$. Defining $\Pi(\phi(t)) = X(t)$, for all $\phi(0) \le t \le 0$ we conclude that $\Pi(\sigma) = P(\sigma)$ for all $\phi(\phi(t)) \le \sigma \le t$, and hence, using (21) we get (24). □

Lemma 3. *There exist a class $\mathcal{KL}$ function β^* and a class $\mathcal{KC}_\infty$ function ρ^* such that for all solutions of the system satisfying (15) for $0 < c < 1$, the following holds:*

$$\Xi(t) \le \beta^*(\rho^*(\Xi(0), c), t), \tag{26}$$

for all $t \ge 0$, where

$$\Xi(t) = |X(t)| + |X(\phi(t))| + \sup_{\phi(\phi(t)) \le \theta \le t} |W(\theta)|. \tag{27}$$

Proof. Based on Assumption 3 and [28], there exist a smooth function $S : \mathbb{R}^n \to \mathbb{R}_+$ and class $\mathcal{K}_\infty$ functions α_4, α_5, α_6 and α_7 such that for all $X \in \mathbb{R}^n$ and for all $\omega \in \mathbb{R}$ the following hold:

$$\alpha_4(|X(\theta)|) \le S(X(\theta)) \le \alpha_5(|X(\theta)|) \tag{28}$$

$$\frac{\partial S(X(\theta))}{\partial X} f(X(\theta), \kappa(X(\theta)) + \omega(\theta)) \le -\alpha_6(|X(\theta)|) + \alpha_7(|\omega(\theta)|). \tag{29}$$

Consider now the following Lyapunov functional for the "target system" given in (22)–(23)

$$V(t) = S(X(t)) + S(X(\phi(t))) + \frac{2 + \frac{1}{1-c}}{g} \int_0^{L(t)} \frac{\alpha_7(r)}{r}\, dr, \tag{30}$$

where

$$L(t) = \sup_{\phi(t) \le \theta \le \sigma(t)} |e^{g(\sigma(\theta)-t)} W(\phi(\theta))| = \lim_{n \to \infty} \left(\int_{\phi(t)}^{\sigma(t)} e^{2ng(\sigma(\theta)-t)} W(\phi(\theta))^{2n}\, d\theta \right)^{1/2n}, \tag{31}$$

with $g > 0$. We now upper- and lower-bound $L(t)$ in terms of $\sup_{\phi(\phi(t)) \le \theta \le t} |W(\theta)|$. From (6), (12) and (15) for $0 < c < 1$ we get for all $\phi(t) \le \theta \le \sigma(t)$ that $d\sigma(\theta)/d\theta = 1/\phi'(\sigma(\theta)) \le 2$. Integrating this relation from $\phi(t)$ to θ and, since $\sigma(\phi(t)) = t$ and $\theta \le \sigma(t)$, we have

$$\sigma(\theta) - t \le 2(\sigma(t) - \phi(t)), \quad \phi(t) \le \theta \le \sigma(t). \tag{32}$$

Since $D \in C^1(\mathbb{R}^n; \mathbb{R}_+)$ there exists a function $\delta_1 \in \mathcal{K}_\infty \cap C^1$ such that

$$D(X) \le D(0) + \delta_1(|X|). \tag{33}$$

Therefore, using (2) and (7) we arrive at

$$L(t) \le e^{4gD(0)} e^{4g\delta_1(|X(t)| + |X(\phi(t))|)} \sup_{\phi(\phi(t)) \le \theta \le t} |W(\theta)|. \tag{34}$$

Moreover, since σ is increasing with $\sigma(\phi(t)) = t$, based on (6) and (15), we get for all $\phi(t) \le \theta \le \sigma(t)$

$$0 \le \sigma(\theta) - t. \tag{35}$$

Therefore, with the help of (35) we have that

$$L(t) \ge \sup_{\phi(\phi(t)) \le \theta \le t} |W(\theta)|. \tag{36}$$

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[2] For computing the first value of $\phi(\theta)$, i.e., $\phi(t + D(X(t)) - h)$, one needs the value of $U(s)$ at $s = \phi(t)$. Since $\phi(t)$ is yet to be computed one could apply a one-discretization step delay h to U and employ the value $U(s)$ at $s = \phi(t-h)$ instead.

[3] The quantities P and Π are identical. However, we use two distinct symbols for the same quantity because, in one case, P is expressed in terms of X and U, for the direct backstepping transformation, while, in the other case, Π is expressed in terms of X and W, for the inverse backstepping transformation.

Taking the time derivative of $L(t)$, with (23) we get

$$\dot{L}(t)=\lim_{n\to\infty}\frac{1}{2n}\left(\int_{\phi(t)}^{\sigma(t)}e^{2ng(\sigma(\theta)-t)}W(\phi(\theta))^{2n}\,d\theta\right)^{1/2n-1}\times\left(-\phi'(t)W(\phi(\phi(t)))^{2n}-2ng\int_{\phi(t)}^{\sigma(t)}e^{2ng(\sigma(\theta)-t)}W(\phi(\theta))^{2n}\,d\theta\right). \tag{37}$$

Using (15) we have $\phi'(t)>0$ and hence $\dot{L}(t)\le -gL(t)$. With this inequality and (29), taking the derivative of (30) we get with the help of (15)

$$\dot{V}(t)\le-\alpha_6(|X(t)|)-\frac{1}{2}\alpha_6(|X(\phi(t))|)+\frac{1}{1-c}\alpha_7(|W(\phi(t))|)+\alpha_7(|W(\phi(\phi(t)))|)-\left(2+\frac{1}{1-c}\right)\alpha_7(L(t)). \tag{38}$$

With the help of (36) we get $\dot{V}(t)\le-\alpha_6(|X(t)|)-\frac{1}{2}\alpha_6(|X(\phi(t))|)-\alpha_7(L(t))$. Using (28), the definition of $L(t)$ in (31) and (30) we conclude that there exists a class $\mathcal{K}$ function γ_1 such that $\dot{V}(t)\le-\gamma_1(V(t))$. Using the comparison principle and Lemma 4.4 in [17], there exists a class $\mathcal{KL}$ function β_1 such that $V(t)\le\beta_1(V(0),t)$. Using (28), the definition of $V(t)$ in (30) and the properties of class $\mathcal{K}$ functions we arrive at $|X(t)|+|X(\phi(t))|+L(t)\le\beta^*(\rho_1(|X(0)|+|X(\phi(0))|+L(0),c),t)$ for some class $\mathcal{KL}$ function β^* and some class $\mathcal{KC}_\infty$ function ρ_1. Using relations (34) and (36) the lemma is proved. □

Lemma 4. *There exists a class $\mathcal{K}_\infty$ function α_8 such that for all solutions of the system satisfying (15) for $0<c<1$, the following holds*:

$$|X(\theta)|+|P(\theta)|\le\alpha_8\left(|X(t)|+|X(\phi(t))|+\sup_{\phi(\phi(t))\le s\le t}|U(s)|\right),\quad \phi(t)\le\theta\le t. \tag{39}$$

Proof. Setting in (4) $\omega=U(\theta)$, we get for all $\phi(t)\le\theta\le t$ that

$$\frac{dR(P(\theta))}{dP}f(P(\theta),U(\theta))\le R(P(\theta))+\alpha_3(|U(\theta)|). \tag{40}$$

Using (7) and (9), by multiplying (40) with $d\sigma(\theta)/d\theta$ and using (15) we get

$$\frac{dR(P(\theta))}{d\theta}\le 2(R(P(\theta))+\alpha_3(|U(\theta)|)),\quad \phi(t)\le\theta\le t. \tag{41}$$

Using the comparison principle and (33) one gets

$$R(P(\theta))\le e^{2(D(0)+\delta_1(|X(\phi(t))|))}\left(R(X(t))+\sup_{\phi(t)\le s\le t}\alpha_3(|U(s)|)\right),\quad \phi(t)\le\theta\le t. \tag{42}$$

Similarly, setting $\omega=U(\phi(\theta))$ in (4) we get for all $\phi(t)\le\theta\le t$

$$\frac{dR(X(\theta))}{dX}f(X(\theta),U(\phi(\theta)))\le R(X(\theta))+\alpha_3(|U(\phi(\theta))|). \tag{43}$$

Since $dX(\theta)/d\theta=f(X(\theta),U(\phi(\theta)))$, for all $\phi(t)\le\theta\le t$, with the comparison principle, (43) and the fact that $t-\phi(t)=D(X(\phi(t)))$ we get

$$R(X(\theta))\le e^{D(0)+\delta_1(|X(\phi(t))|)}\left(R(X(\phi(t)))+\sup_{\phi(\phi(t))\le s\le\phi(t)}\alpha_3(|U(s)|)\right). \tag{44}$$

With standard properties of class $\mathcal{K}_\infty$ functions we get (39), where the class $\mathcal{K}_\infty$ function α_8 is given as

$$\alpha_8(s)=2\alpha_1^{-1}((\alpha_2(s)+\alpha_3(s))e^{2(D(0)+\delta_1(s))}).\qquad\square \tag{45}$$

Lemma 5. *There exists a class $\mathcal{K}$ function γ^* such that for all solutions of the system satisfying (15) for $0<c<1$, the following holds*:

$$|X(\theta)|+|\Pi(\theta)|\le\gamma^*(|X(t)|+|X(\phi(t))|+\sup_{\phi(\phi(t))\le s\le t}|W(s)|),\quad \phi(t)\le\theta\le t. \tag{46}$$

Proof. Under Assumption 3 and [28], there exists class $\mathcal{KL}$ function β_2 and class $\mathcal{K}$ function γ_1 such that

$$|Y(\tau)|\le\beta_2(|Y(t_0)|,\tau-t_0)+\gamma_1\left(\sup_{t_0\le s\le\tau}|\omega(s)|\right),\quad \tau\ge t_0, \tag{47}$$

where $Y(\tau)$ is the solution of $\dot{Y}(\tau)=f(Y(\tau),\kappa(Y(\tau))+\omega(\tau))$. Using the change of variables $y=\sigma(\theta)$ and (7), (25), we have that

$$\frac{d\Pi(\phi(y))}{dy}=f(\Pi(\phi(y)),\kappa(y,\Pi(\phi(y)))+W(\phi(y))). \tag{48}$$

Using (47) we have

$$|\Pi(\theta)|\le\gamma_2(|X(t)|)+\gamma_1\left(\sup_{\phi(t)\le s\le t}|W(s)|\right),\quad \phi(t)\le\theta\le t, \tag{49}$$

where the class $\mathcal{K}$ function γ_2 is defined as $\gamma_2(s)=\beta_2(s,0)$. Analogously, since $X(\theta)$ satisfies $dX(\theta)/d\theta=f(X(\theta),\kappa(X(\theta))+W(\phi(\theta)))$, for all $\phi(t)\le\theta\le t$, we get

$$|X(\theta)|\le\gamma_2(|X(\phi(t))|)+\gamma_1\left(\sup_{\phi(\phi(t))\le s\le\phi(t)}|W(s)|\right),\quad \phi(t)\le\theta\le t. \tag{50}$$

With the properties of class $\mathcal{K}$ functions we get (46), where $\gamma^*(s)=2\gamma_1(s)+2\gamma_2(s)$ is of class $\mathcal{K}$. □

Lemma 6. *There exist class $\mathcal{K}_\infty$ functions α_9 and α_{10} such that for all solutions of the system satisfying (15) for $0<c<1$, the following hold*:

$$\Omega(t)\le\alpha_9(\Xi(t)) \tag{51}$$

$$\Xi(t)\le\alpha_{10}(\Omega(t)), \tag{52}$$

for all $t\ge0$ where Ω is defined in (17) and Ξ is defined in (27).

Proof. Using (46) we have that

$$\sup_{\phi(t)\le\theta\le t}|X(\theta)|\le\gamma^*\left(|X(t)|+|X(\phi(t))|+\sup_{\phi(\phi(t))\le s\le t}|W(s)|\right). \tag{53}$$

Under Assumption 3 (Lipschitzness of κ and $\kappa(0)=0$) there exists a class $\mathcal{K}_\infty$ function $\hat{\alpha}$ such that

$$|\kappa(X)|\le\hat{\alpha}(|X|). \tag{54}$$

With the inverse backstepping transformation (24) and relation (54) we arrive at

$$\sup_{\phi(\phi(t))\le\theta\le t}|U(\theta)|\le\sup_{\phi(\phi(t))\le\theta\le t}|W(\theta)|+\hat{\alpha}\left(\sup_{\phi(\phi(t))\le\theta\le\phi(t)}|\Pi(\theta)|+\sup_{\phi(t)\le\theta\le t}|\Pi(\theta)|\right). \tag{55}$$

Using relation (46) and definition $\Pi(\phi(\theta))=X(\theta)$, $\phi(t)\le\theta\le t$ we get (51) with

$$\alpha_9(s)=s+\gamma^*(s)+\hat{\alpha}(2\gamma^*(s)). \tag{56}$$

Analogously, using the direct backstepping transformation (21), relation (39) and definition $P(\phi(\theta))=X(\theta)$, $\phi(t)\le\theta\le t$ we get (52) with $\alpha_{10}(s)=s+\hat{\alpha}(2\alpha_8(s))$. □

Lemma 7. *There exists a function α^* of class $\mathcal{K}_\infty$ such that all the solutions that satisfy*

$$\Omega(t)<\alpha^{*-1}(c),\quad t\ge0, \tag{57}$$

for $0<c<1$ also satisfy (15) where Ω is defined in (17).

Proof. Since $D \in C^1(\mathbb{R}^n; \mathbb{R}_+)$ and $f: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is locally Lipschitz with $f(0,0)=0$, there exist class $\mathcal{K}_\infty$ functions δ_2 and α_{11} such that

$$|\nabla D(X)| \le |\nabla D(0)| + \delta_2(|X|) \tag{58}$$

$$|f(X,\omega)| \le \alpha_{11}(|X| + |\omega|). \tag{59}$$

If a solution satisfies for all $t \ge 0$

$$(|\nabla D(0)| + \delta_2(|X(\theta)|))\alpha_{11}(|X(\theta)| + |U(\phi(\theta))|) < c, \quad \phi(t) \le \theta \le t, \tag{60}$$

for $0 < c < 1$, then it also satisfies (15). With the trivial inequalities $|X(\theta)| \le \sup_{\phi(t) \le \tau \le t}|X(\tau)|$ and $|U(\phi(\theta))| \le \sup_{\phi(\phi(t)) \le s \le \phi(t)}|U(s)|$, for all $\phi(t) \le \theta \le t$, relation (60) is satisfied for $0 < c < 1$, for all $t \ge 0$, as long as (57) holds, where the class $\mathcal{K}_\infty$ function α^* is defined as

$$\alpha^*(s) = (|\nabla D(0)| + \delta_2(s))\alpha_{11}(s). \quad \square \tag{61}$$

Lemma 8. *There exists a class $\mathcal{K}$ function ψ_{RoA} such that for all initial conditions of the closed-loop system (1), (10), (12), (13) that satisfy relation (16) the solutions of the system satisfy (57) for $0 < c < 1$ and hence satisfy (15).*

Proof. Using Lemma 6, with the help of (26) we have that

$$\Omega(t) \le \alpha_9(\beta^*(\rho^*(\alpha_{10}(\Omega(0)), c), t)). \tag{62}$$

By defining the class $\mathcal{K}_\infty$ function α_{12} as $\alpha_{12}(s) = \alpha_9(\beta^*(s,0))$, we get

$$\Omega(t) \le \alpha_{12}(\rho^*(\alpha_{10}(\Omega(0)), c)). \tag{63}$$

Hence, for all initial conditions that satisfy the bound (16) with any choice of a class $\mathcal{K}$ function $\psi_{\mathrm{RoA}}(c) \le \overline{\psi^*}_{\mathrm{RoA}}(\alpha^{*-1}(c), c)$, where $\overline{\psi^*}_{\mathrm{RoA}}(s,c)$ is the inverse of the class $\mathcal{KC}_\infty$ function $\psi^*_{\mathrm{RoA}}(s,c) = \alpha_{12}(\rho^*(\alpha_{10}(s), c))$ with respect to ψ^*_{RoA}'s first argument, the solutions satisfy (57). Furthermore, for all those initial conditions, the solutions verify (15) for all $\sigma \ge 0$. $\square$

Proof of Theorem 1. Using (62) we get (18) with $\beta(s,t) = \alpha_9(\beta^*(s,t))$ and $\rho(s,c) = \rho^*(\alpha_{10}(s), c)$. System (22), (23) guarantees the existence and uniqueness of $X \in C^1(\sigma^*, \infty)$, where $\sigma^* = D(X(0))$. Consider now the case $t \in [0, \sigma^*)$. From (1) and (11) we have for all $t \in [0, \sigma^*)$ that

$$\dot{X}(t) = f(X(t), g_U(\phi(t))) \tag{64}$$

$$\dot{\phi}(t) = \frac{1}{1 + \nabla D(g_X(\phi(t)))f(g_X(\phi(t)), g_{U_d}(\phi(t)))}, \tag{65}$$

where the initial conditions for X, U and $U(\phi)$ are defined as $X(s) = g_X(s)$, $U(s) = g_U(s)$ and $U(\phi(s)) = g_{U_d}(s)$ for all $\phi(0) \le s < 0$. Lipschitzness of g_X, g_U, g_{U_d} on $[\phi(0), 0)$, Lipschitzness of f and Assumption 1 (Lipschitzness of ∇D) guarantee that the right-hand side of (64) and of (65) is Lipschitz with respect to (X, ϕ), which guarantees, together with bound (15), the existence and uniqueness of $X \in C^1[0, \sigma^*)$. The boundedness of W and (22) guarantee that X is continuous at $t = \sigma^*$. By integrating (22) between any two time instants it is shown that X is Lipschitz on $[0, \infty)$ with a Lipschitz constant given by a uniform bound on the right-hand side of (22). The fact that $\Pi(t) = X(t + D(X(t)))$ for all $t \ge 0$ and Assumption 1 ($D \in C^1(\mathbb{R}^n; \mathbb{R}_+)$) guarantee that Π is Lipschitz for all $t \ge 0$. Since $U(t) = \kappa(\Pi(t))$, Assumption 3 (Lipschitzness of κ in both arguments) guarantees that U is Lipschitz in t on $(0, \infty)$. Using (15) and (33), we get (19), (20) with any class $\mathcal{K}$ function $\delta^*(c) \ge \delta_1(\alpha^{*-1}(c))$. $\square$

4. Example

In this example we consider the following system in the feedforward form taken from [18]

$$\dot{X}_1(t) = X_2(t) - X_2(t)^2 U(t - D(X(\phi(t)))) \tag{66}$$

$$\dot{X}_2(t) = U(t - D(X(\phi(t)))), \tag{67}$$

where

$$\phi(t) = t - D(X(\phi(t))), \tag{68}$$

$$D(X(\phi(t))) = \tfrac{1}{2}\sin(5X_2(\phi(t)))^2, \tag{69}$$

and hence,

$$\sigma(t) = t + \tfrac{1}{2}\sin(5X_2(t))^2. \tag{70}$$

A nominal design for the delay-free plant is given in [18] as

$$U(t) = -X_1(t) - 2X_2(t) - \tfrac{1}{3}X_2(t)^2. \tag{71}$$

The predictor-based design is

$$U(t) = -P_1(t) - 2P_2(t) - \tfrac{1}{3}P_2(t)^2, \tag{72}$$

where

$$P_1(t) = X_1(t) + \int_{\phi(t)}^{t}(1 + 5\cos(5X_2(s))\sin(5X_2(s))U(\phi(s)))(P_2(\theta) - P_2(\theta)^2 U(\theta))\, d\theta \tag{73}$$

$$P_2(t) = X_2(t) + \int_{\phi(t)}^{t}(1 + 5\cos(5X_2(s))\sin(5X_2(s))U(\phi(s)))U(\theta)\, d\theta, \tag{74}$$

where

$$\phi(t) = t - \int_{t}^{t+1/2\sin(5X_2(t))^2} \frac{d\theta}{1 + 5\cos(5X_2(\phi(s)))\sin(5X_2(\phi(s)))U(\phi(\phi(s)))}. \tag{75}$$

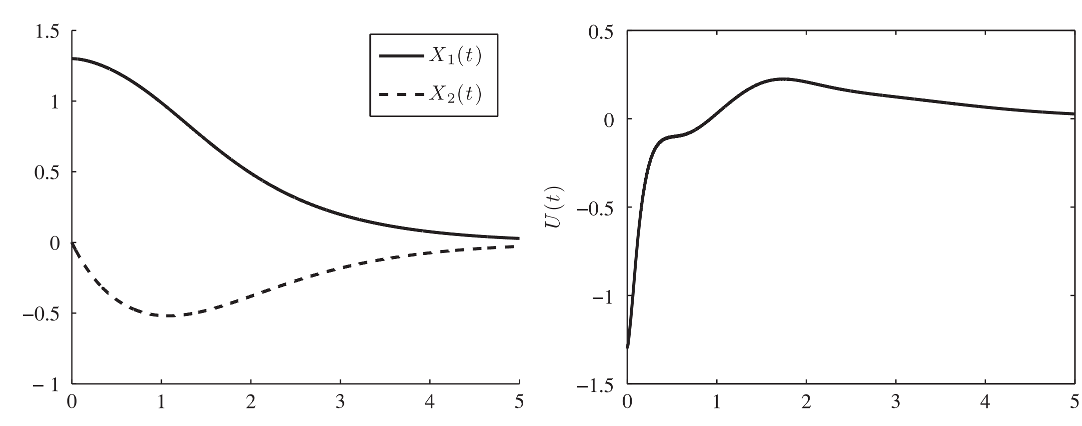

Fig. 1. The response of the closed-loop system (66), (67) with the predictor feedback law (72)–(75) (left) and the control effort (72) (right). The initial conditions are chosen as $X_1(0) = 1.3$, $X_2(s) = 0$ for all $\phi(0) \le s \le 0$ and $U(s) = U(\phi(s)) = 0$ for all $\phi(0) \le s \le 0$. For these initial conditions the control signal "kicks in" at $t = 0$, and hence, the delay is immediately compensated, resulting to identical, to the delay-free case, responses for X_1, X_2. The control signal oscillates to compensate the oscillatory delay.

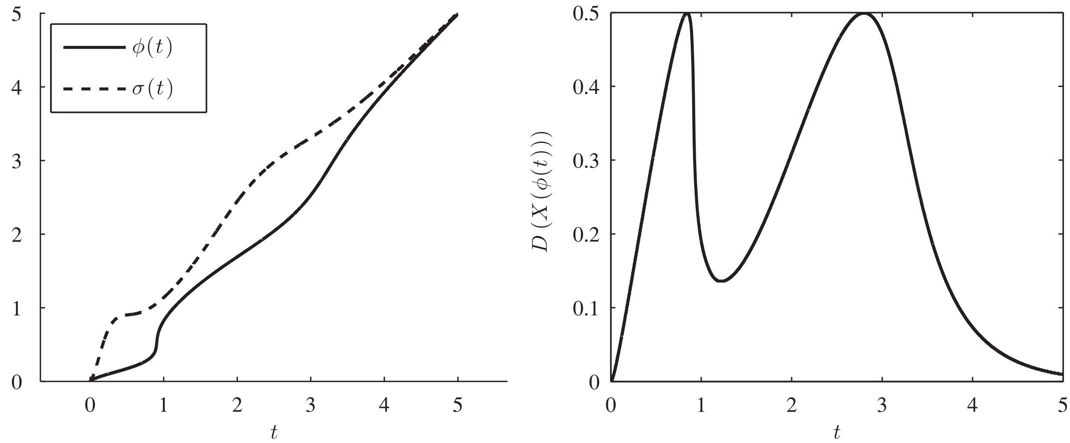

Fig. 2. The delayed time (68) and the prediction time (70) (left), and the delay (69) (right), of the closed-loop system (66), (67) with the predictor feedback law (72)–(75). The initial conditions are chosen as $X_1(0)=1.3$, $X_2(s)=0$ for all $\phi(0)\le s\le 0$ and $U(s)=U(\phi(s))=0$ for all $\phi(0)\le s\le 0$. For these initial conditions $\phi(0)=\sigma(0)=0$.

We consider the initial conditions for the plant as $X_1(0)=1.3$, $X_2(s)=0$ for all $\phi(0)\le s\le 0$, and the initial conditions for the actuator state as $U(s)=U(\phi(s))=0$ for all $\phi(0)\le s\le 0$. With such an initial condition we get $\phi(0)=0$. Therefore, the control signal "kicks in" immediately, i.e., at $t=0$. In Fig. 1 we show the response of the closed-loop system. As one can observe that the delay is immediately compensated and the responses of X_1 and X_2 are as there were no delay at all. Yet, the oscillations are evident in the control signal. The control signal oscillates in order to compensate the oscillatory delay. In Fig. 2 we show the delayed time ϕ and the prediction time σ.

5. Stabilization under state delays

In the present section we consider the following plant:

$$\dot{X}(t)=f(X(t),\zeta(\phi(t))) \tag{76}$$

$$\dot{\zeta}(t)=U(t), \tag{77}$$

where $t\ge 0$, $U,\zeta\in\mathbb{R}$, $f:\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}^n$ is locally Lipschitz with $f(0,0)=0$ and ϕ satisfies

$$t=\phi(t)+D(\zeta(\phi(t))). \tag{78}$$

Remark 1. Let us highlight the importance of considering the class of nonlinear systems that satisfy (76)–(78). Firstly, this is a class of systems with state delay that depends on the past state of the system. This is different than system (1) which has an input rather a state delay. Secondly, and more importantly, when one stabilizes system (76)–(78) then one can stabilize nonlinear systems with input delays that depends on the past input rather than the past state. To see this consider a nonlinear system with input delay that depends on past values of the input, i.e., consider the system

$$\dot{X}(t)=f(X(t),V(\phi(t))), \tag{79}$$

where $t=\phi(t)+D(V(\phi(t)))$. Then, by adding an integrator, one gets exactly Eqs. (76)–(78) with $\zeta=V$ and $\dot{V}=U$, where U is designed in order to stabilize the system $\dot{X}(t)=f(X(t),V(\phi(t)))$, $\dot{V}(t)=U(t)$. Hence, stabilization of system (76), (77), (78) implies stabilization of system (79).

We make now the following assumption regarding system (76).

Assumption 4. There exists a function $\mu\in C^1(\mathbb{R}^n;\mathbb{R})$, with $\mu(0)=0$ and $\nabla\mu$ locally Lipschitz,[4] such that the plant $\dot{X}(t)=f(X(t),\mu(X(t))+\omega(t))$ is input-to-state stable with respect to ω.

[4] To ensure uniqueness of solutions.

Note that we still assume that the delay function D (which is now defined in $\mathbb{R}$ rather than in $\mathbb{R}^n$) and the vector field f satisfy Assumptions 1 and 2 respectively. Assumption 4 is similar to Assumption 3 with the difference that in the present case the feedback law μ is assumed continuously differentiable rather than just locally Lipschitz (this regularity requirement for μ is a result of the backstepping procedure). Finally, note that the results of this section can be extended to the case at which the delay (78) depends also on $X(\phi(t))$. However, in order to keep the formulae of our design as simple as possible we do not consider this case.

From plant (76), (77) one can observe that the input signal reaches the state ζ at $t=0$. However, it reaches X through a delayed integrator. Therefore, we need to define and derive an implementable form for the predictor of the state X, i.e., the signal that satisfies $P(\phi(t))=X(t)$ for all $t\ge 0$. From relation (78) we get that

$$\begin{aligned}\phi^{-1}(\theta)&=\sigma(\theta)\\&=\theta+D(\zeta(\theta))\quad\text{for all } \phi(t)\le\theta\le t.\end{aligned} \tag{80}$$

Setting $t=\sigma(\theta)$ in (76), differentiating with respect to θ and integrating the resulting expression from $\phi(t)$ to θ, with the help of the fact that $P(\phi(t))=X(t)$ we get

$$P(\theta)=X(t)+\int_{\phi(t)}^{\theta}(1+D'(\zeta(s))U(s))f(P(s),\zeta(s))\,ds$$
$$\text{for all } \phi(t)\le\theta\le t. \tag{81}$$

We compute next ϕ. Differentiating (78) and since $\phi(t+D(\zeta(t)))=\phi(\sigma(t))=t$ we get that

$$\phi(\theta)=t-\int_{\theta}^{t+D(\zeta(t))}\frac{ds}{1+D'(\zeta(\phi(s)))U(\phi(s))}\quad\text{for all } t\le\theta\le\sigma(t). \tag{82}$$

The predictor-based control law is based on a backstepping design on the delay plant and is given by

$$U(t)=\frac{\nabla\mu(P(t))f(P(t),\zeta(t))-c_Z(\zeta(t)-\mu(P(t)))}{1-\nabla\mu(P(t))f(P(t),\zeta(t))D'(\zeta(t))}, \tag{83}$$

where $c_Z>0$ is arbitrary. Analogously to case of input delay, in an actual implementation of the control law (83), (81), (82) one has to compute, at each time step, $\phi(t)$, by numerically computing the integral in (82) and using the history of ζ and U. Then, one computes $P(t)$ using $\phi(t)$ and the history of ζ, P and U. Finally, one calculates $U(t)$ from (83). However, in order to compute $\phi(t)$ one starts the integration at $\sigma(t)=t+D(\zeta(t))$. Yet, the function inside the integral evaluated at $s=\sigma(t)$ depends on $U(t)$, i.e., on the current value of the input, which is yet to be computed. Therefore, since $\sigma(t)$ is strictly increasing one can compute $\phi(t)$ by integrating (82) up to $s=\sigma(t-h)$, where h is the discetization step.

From (83) one can observe that besides a restriction that the denominator in (82) is positive, one has an additional condition that the denominator in (83) is also positive. Both conditions are satisfied when the following condition holds:

$$\mathcal{G}_c : |D'(\zeta(\theta))U(\theta)| + |\nabla\mu(P(\theta))f(P(\theta),\zeta(\theta))D'(\zeta(\theta))| < c \quad \text{for all } \theta \geq \phi(0), \tag{84}$$

for some $0 < c < 1$.

Theorem 2. *Consider the plant (76)–(78) together with the control law (83), (81), (82). Under Assumptions 1, 2 and 4 there exist a class $\mathcal{K}$ function ξ_{RoA}, a class $\mathcal{KL}$ function $\hat{\beta}$ and a class $\mathcal{K}_\infty$ function σ_1 such that for all initial conditions for which ζ is locally Lipschitz on the interval $[\phi(0),0]$, U is locally Lipschitz on the interval $[\phi(0),0)$, and they satisfy (77) and*

$$\hat{\Omega}(0) < \xi_{\mathrm{RoA}}(c), \tag{85}$$

where

$$\hat{\Omega}(t) = |X(t)| + \sup_{\phi(t)\leq\theta\leq t}|\zeta(\theta)| + \sup_{\phi(t)\leq\theta\leq t}|U(\theta)|, \tag{86}$$

for some $0 < c < 1$, there exists a unique solution to the closed-loop system with $X \in C^1[0,\infty)$, ζ Lipschitz on $[0,\infty)$, U Lipschitz on $(0,\infty)$ and

$$\hat{\Omega}(t) \leq \sigma_1\left(1+\frac{1}{1-c}\right)\hat{\beta}(\hat{\Omega}(0),t), \tag{87}$$

for all $t \geq 0$. Furthermore, there exists a class $\mathcal{K}$ function $\hat{\delta}^*$, such that for all $t \geq 0$ the following hold:

$$\sup_{\phi(t)\leq\theta\leq t} D(\zeta(\theta)) \leq D(0) + \hat{\delta}^*(c) \tag{88}$$

$$\sup_{\phi(t)\leq\theta\leq t} |\dot{D}(\zeta(\theta))| \leq c. \tag{89}$$

The proof of Theorem 2 is based on Lemmas 9–16 which are presented next.

Lemma 9. *The infinite-dimensional backstepping transformation of the state ζ defined by*

$$Z(\theta) = \zeta(\theta) - \mu(P(\theta)), \quad \phi(t) \leq \theta \leq t, \tag{90}$$

together with the predictor-based control law (83), (81), (82) transform the system (76)–(77) to the "target system" given by

$$\dot{X}(t) = f(X(t), \mu(X(t)) + Z(\phi(t))) \tag{91}$$

$$\dot{Z}(t) = -c_Z Z(t). \tag{92}$$

Proof. Using (76) and the fact that $P(\phi(t)) = X(t)$ we get (91). Setting $\theta = t$ in (90) and taking the derivative with respect to t of the resulting equation we get (92) using (77), (81) and (83). □

Lemma 10. *The inverse of the infinite-dimensional backstepping transformation defined in (90) is*

$$\zeta(\theta) = Z(\theta) + \mu(\Pi(\theta)), \quad \phi(t) \leq \theta \leq t, \tag{93}$$

where

$$\Pi(\theta) = X(t) + \int_{\phi(t)}^{\theta}(1 + D'(\mu(\Pi(s)) + Z(s))U(s)) \times f(\Pi(s), \mu(\Pi(s)) + Z(s))\, ds, \quad \phi(t) \leq \theta \leq t. \tag{94}$$

Proof. By direct verification, noting also that $\Pi(\theta) = P(\theta)$ for all $\phi(t) \leq \theta \leq t$, where $\Pi(\theta)$ is driven by the transformed state $Z(\theta)$, whereas $P(\theta)$ is driven by the state $\zeta(\theta)$ for $\phi(t) \leq \theta \leq t$. □

Lemma 11. *There exists a class $\mathcal{KL}$ function $\hat{\beta}^*$ such that for all solutions of the system satisfying (84) for $0 < c < 1$, the following holds for all $t \geq 0$:*

$$\hat{\Xi}(t) \leq \left(1+\frac{1}{1-c}\right)(\hat{\beta}^*(\hat{\Xi}(0),t) + \hat{\beta}_4(\hat{\Xi}(0), \max\{\{0, t-\sigma(0)\}\})), \tag{95}$$

where

$$\hat{\Xi}(t) = |X(t)| + \sup_{\phi(t)\leq\theta\leq t}|Z(\theta)| + \sup_{\phi(t)\leq\theta\leq t}|U(\theta)|. \tag{96}$$

Proof. Solving (92), we have that $Z(t) = Z(0)e^{-c_Z(t)}$ for all $t \geq 0$. Since $\phi(t)$ is increasing for all $t \geq 0$ we get

$$\sup_{\phi(t)\leq\theta\leq t}|Z(\theta)| \leq |Z(0)|e^{-c_Z\phi(t)} \quad \text{for all } t \geq \sigma(0). \tag{97}$$

Similarly, for all $0 \leq t \leq \sigma(0)$ we get

$$\sup_{\phi(t)\leq\theta\leq t}|Z(\theta)| \leq \sup_{\phi(0)\leq\theta\leq 0}|Z(\theta)| + \sup_{0\leq\theta\leq t\leq\sigma(0)}|Z(\theta)|, \tag{98}$$

and hence, combining (98) with (92), we get

$$\sup_{\phi(t)\leq\theta\leq t}|Z(\theta)| \leq 2\sup_{\phi(t_0)\leq\theta\leq t_0}|Z(\theta)| \quad \text{for all } 0 \leq t \leq \sigma(0). \tag{99}$$

Therefore, using (97), (99) and the fact that for all $t \leq \sigma(0)$, $\phi(t) \leq 0$ we get

$$\sup_{\phi(t)\leq\theta\leq t}|Z(\theta)| \leq 2\sup_{\phi(0)\leq\theta\leq 0}|Z(\theta)|e^{-c_Z\phi(t)} \quad \text{for all } t \geq 0. \tag{100}$$

Using (90) we get that $\phi(t) = t - D(\zeta(\phi(t))) = t - D(Z(\phi(t)) + \mu(X(t)))$, and hence,

$$\sup_{\phi(t)\leq\theta\leq t}|Z(\theta)| \leq 2\sup_{\phi(0)\leq\theta\leq 0}|Z(\theta)|e^{-c_Z t}e^{c_Z D(Z(\phi(t))+\mu(X(t)))} \quad \text{for all } t \geq 0. \tag{101}$$

Using (33) we get that $D(Z(\phi(t)) + \mu(X(t))) \leq D(0) + \delta_1(2|Z(\phi(t))|) + \delta_1(2|X(t)|)$. Since for all $t \geq \sigma(0)$, $\phi(t) \geq 0$, from (92) we get that $|Z(\phi(t))| \leq |Z(0)|$ for all $t \geq \sigma(0)$. Moreover, for all $t \leq \sigma(0)$, $\phi(0) \leq \phi(t) \leq 0$. Hence, for all $t \geq 0$, $|Z(\phi(t))| \leq \sup_{\phi(0)\leq\theta\leq 0}|Z(\theta)|$. Therefore, (101) gives

$$\sup_{\phi(t)\leq\theta\leq t}|Z(\theta)| \leq 2\sup_{\phi(0)\leq\theta\leq 0}|Z(\theta)|e^{c_Z(D(0)+\delta_1(2\sup_{\phi(0)\leq\theta\leq 0}|Z(\theta)|)+\delta_1(2|X(t)|))}e^{-c_Z t} \quad \text{for all } t \geq 0. \tag{102}$$

Let $Y(s)$ be the solution of $dY(s)/ds = f(Y(s), \mu(Y(s)) + \omega(s))$ for $s \geq s_0 \geq 0$. Under Assumption 4 and [28], there exist class $\mathcal{KL}$ function $\hat{\beta}_2$ and class $\mathcal{K}$ function $\hat{\gamma}_1$ such that

$$|Y(s)| \leq \hat{\beta}_2(|Y(s_0)|, s-s_0) + \hat{\gamma}_1\left(\sup_{s_0\leq r\leq s}|\omega(r)|\right) \quad \text{for all } s \geq s_0, \tag{103}$$

and hence, with (91) we get

$$|X(t)| \leq \hat{\beta}_2(|X(s)|, t-s) + \hat{\gamma}_1\left(\sup_{s\leq\tau\leq t}|Z(\phi(\tau))|\right) \quad \text{for all } t \geq s \geq 0. \tag{104}$$

Setting $s = 0$ we have that

$$|X(t)| \leq \hat{\beta}_2(|X(0)|, t) + \hat{\gamma}_1\left(\sup_{\phi(0)\leq\theta\leq\phi(t)}|Z(\theta)|\right) \quad \text{for all } t \geq 0, \tag{105}$$

and hence, from (92)

$$|X(t)| \leq \hat{\beta}_2(|X(0)|, 0) + \hat{\gamma}_1\left(2\sup_{\phi(0)\leq\theta\leq 0}|Z(\theta)|\right) \quad \text{for all } t \geq 0. \tag{106}$$

Therefore, with (102) we arrive at

$$\sup_{\phi(t)\leq\theta\leq t}|Z(\theta)| \leq \hat{\alpha}_{12}\left(|X(0)| + \sup_{\phi(0)\leq\theta\leq 0}|Z(\theta)|\right)e^{-c_Z t} \quad \text{for all } t \geq 0, \tag{107}$$

where the class $\mathcal{K}_\infty$ function $\hat{\alpha}_{12}$ is defined as

$$\hat{\alpha}_{12}(s) = 2e^{c_Z D(0)}se^{c_Z(\delta_1(s)+\delta_1(2\hat{\beta}_2(s,0)+2\hat{\gamma}_1(s)))}. \tag{108}$$

Setting in (104) $s=t/2$ we get

$$|X(t)| \le \hat{\beta}_2\left(|X(0)|, \frac{t}{2}\right) + \hat{\gamma}_1\left(\sup_{\phi(\frac{t}{2}) \le \theta \le \phi(t)} |Z(\theta)|\right) \quad \text{for all } t \ge 0. \tag{109}$$

We estimate now $\sup_{\phi(t/2) \le \theta \le \phi(t)} |Z(\theta)|$. Solving (92) we get

$$\sup_{\phi(t/2) \le \theta \le \phi(t)} |Z(\theta)| \le 2 \sup_{\phi(0) \le \theta \le 0} |Z(\theta)| e^{-c_Z \phi(t/2)} \quad \text{for all } t \ge 2\sigma(0). \tag{110}$$

With the help of relations (92) and (110) we get

$$\sup_{\phi(t/2) \le \theta \le \phi(t)} |Z(\theta)| \le \sup_{\phi(t/2) \le \theta \le 0} |Z(\theta)| + \sup_{0 \le \theta \le \phi(t)} |Z(\theta)| \le 2 \sup_{\phi(0) \le \theta \le 0} |Z(\theta)|$$

$$\text{for all } 0 \le t \le 2\sigma(0). \tag{111}$$

Hence, using the fact that $\phi(t/2) = t/2 - D(\zeta(\phi(t/2)))$ we get from (90)

$$\sup_{\phi(t/2) \le \theta \le \phi(t)} |Z(\theta)| \le 2 \sup_{\phi(0) \le \theta \le 0} |Z(\theta)| e^{c_Z(D(0)+\delta_1(2|X(t/2)|)+\delta_1(2|Z(\phi(t/2))|))} e^{-(c_Z/2)t}, \tag{112}$$

for all $t \ge 0$. Setting $s=0$ and replacing t by $t/2$ we get from (104) that

$$\left|X\left(\frac{t}{2}\right)\right| \le \hat{\beta}_2\left(|X_1(0)|, \frac{t}{2}\right) + \hat{\gamma}_1\left(\sup_{\phi(0) \le \theta \le \phi(t/2)} |Z(\theta)|\right). \tag{113}$$

Since, $\sup_{\phi(0) \le \theta \le \phi(t/2)} |Z(\theta)| \le \sup_{\phi(0) \le \theta \le 0} |Z(\theta)| + \sup_{0 \le \theta \le \phi(t)} |Z(\theta)| \le 2 \sup_{\phi(0) \le \theta \le 0} |Z(\theta)|$, we get

$$\left|X\left(\frac{t}{2}\right)\right| \le \hat{\beta}_2(|X(0)|, 0) + \hat{\gamma}_1\left(2 \sup_{\phi(0) \le \theta \le 0} |Z(\theta)|\right) \quad \text{for all } t \ge 0. \tag{114}$$

Using also the fact that $|Z(\phi(t/2))| \le \sup_{\phi(0) \le \theta \le 0} |Z(\theta)|$, combining (112), (114) we get from (109) that

$$|X(t)| \le \hat{\beta}_3\left(|X(0)| + \sup_{\phi(0) \le \theta \le 0} |Z(\theta)|, t\right) \quad \text{for all } t \ge 0 \tag{115}$$

where the class $\mathcal{KL}$ function $\hat{\beta}_3$ is defined as

$$\hat{\beta}_3(s,t) = \hat{\beta}_2\left(s, \frac{t}{2}\right) + \hat{\gamma}_1(2s e^{c_Z D(0)} e^{c_Z(\delta_1(2s)+\delta_1(2\hat{\beta}_2(s,0)+2\hat{\gamma}_1(s)))} e^{-c_Z t/2}). \tag{116}$$

Using (83), (84) we get for all $\theta \ge 0$

$$|U(\theta)| \le \frac{1}{1-c} |\nabla \mu(P(\theta)) f(P(\theta), \zeta(\theta))| + \frac{1}{1-c} c_Z |Z(\theta)|. \tag{117}$$

Since $f: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is locally Lipschitz with $f(0,0)=0$ and $\mu \in C^1(\mathbb{R}^n; \mathbb{R})$ with $\mu(0)=0$, there exist class $\mathcal{K}_\infty$ functions $\hat{\alpha}_{11}$ and $\hat{\alpha}_{11}^*$ such that for all $(X, \omega) \in \mathbb{R}^{n+1}$

$$|\mu(X)| \le \hat{\alpha}_{11}(|X|) \tag{118}$$

$$|\nabla \mu(X)| \le |\nabla \mu(0)| + \hat{\alpha}_{11}(|X|) \tag{119}$$

$$|f(X, \omega)| \le \hat{\alpha}_{11}^*(|X| + |\omega|). \tag{120}$$

Therefore, using (33), (90) and the fact that $P(\theta) = \Pi(\theta) = X(\sigma(\theta))$ we get for all $\theta \ge 0$ that

$$|U(\theta)| \le \frac{1}{1-c}(|\nabla \mu(0)| + \hat{\alpha}_{11}(|X(\sigma(\theta))|)) \hat{\alpha}_{11}^*(|X(\sigma(\theta))| + |Z(\theta)| + \hat{\alpha}_{11}(|X(\sigma(\theta))|)) + \frac{1}{1-c} c_Z |Z(\theta)|. \tag{121}$$

Hence, with the help of (107), (115)

$$\sup_{\phi(t) \le \theta \le t} |U(\theta)| \le \frac{1}{1-c} \hat{\beta}_4\left(|X(0)| + \sup_{\phi(0) \le \theta \le 0} |Z(\theta)|, t\right) \quad \text{for all } t \ge \sigma(0), \tag{122}$$

where the class $\mathcal{KL}$ function $\hat{\beta}_4$ is defined as

$$\hat{\beta}_4(s,t) = (|\nabla \mu(0)| + \hat{\alpha}_{11}(\hat{\beta}_3(s,t))) \hat{\alpha}_{11}^*(\hat{\beta}_3(s,t) + \hat{\alpha}_{12}(s) e^{-c_Z t} + \hat{\alpha}_{11}(\hat{\beta}_3(s,t))) + c_Z \hat{\alpha}_{12}(s) e^{-c_Z t}. \tag{123}$$

Moreover, $\sup_{\phi(t) \le \theta \le t} |U(\theta)| \le \sup_{\phi(0) \le \theta \le 0} |U(\theta)| + \sup_{0 \le \theta \le t} |U(\theta)|$, for all $t \le \sigma(0)$, and hence,

$$\sup_{\phi(t) \le \theta \le t} |U(\theta)| \le \sup_{\phi(0) \le \theta \le 0} |U(\theta)| + \frac{1}{1-c} \hat{\beta}_4\left(|X(0)| + \sup_{\phi(0) \le \theta \le 0} |Z(\theta)|, 0\right) \quad \text{for all } t \le \sigma(0). \tag{124}$$

Combining (122), (124) and assuming without loss of generality that $\hat{\beta}_4(s,0) > s$ we arrive at

$$\sup_{\phi(t) \le \theta \le t} |U(\theta)| \le \left(1 + \frac{1}{1-c}\right) \hat{\beta}_4(\hat{\Xi}(0), \max\{\{0, t-\sigma(0)\}\}) \quad \text{for all } t \ge 0. \tag{125}$$

Combining (107), (115), (125) we get (95) with $\hat{\beta}^*(s,t) = \hat{\alpha}_{12}(s) e^{-c_Z t} + \hat{\beta}_3(s,t)$. □

Lemma 12. *There exists a class $\mathcal{K}_\infty$ function $\hat{\alpha}_8$ such that for all solutions of the system satisfying (84) for $0 < c < 1$, the following holds*:

$$|P(\theta)| \le \hat{\alpha}_8\left(|X(t)| + \sup_{\phi(t) \le \tau \le t} |\zeta(\tau)|\right), \quad \phi(t) \le \theta \le t. \tag{126}$$

Proof. Under Assumption 2 we have that

$$\frac{dR(P(\theta))}{dP} f(P(\theta), \zeta(\theta)) \le R(P(\theta)) + \alpha_3(|\zeta(\theta)|), \quad \phi(t) \le \theta \le t. \tag{127}$$

Multiplying both sides of (127) with $\dot{\sigma}(\theta) = 1 + D'(\zeta(\theta)) U(\theta) > 0$, with (84) we get that

$$\frac{dR(P(\theta))}{d\theta} \le 2(R(P(\theta)) + \alpha_3(|\zeta(\theta)|)), \quad \phi(t) \le \theta \le t. \tag{128}$$

Using relation (33) and the comparison principle we have from (127) for all $\phi(t) \le \theta \le t$ that

$$R(P(\theta)) \le e^{2(D(0)+\delta_1(|\zeta(\phi(t))|))}\left(R(X(t)) + \sup_{\phi(t) \le \tau \le t} \alpha_3(|\zeta(\tau)|)\right). \tag{129}$$

With standard properties of class $\mathcal{K}_\infty$ functions we get the statement of the lemma with $\hat{\alpha}_8 \in \mathcal{K}_\infty$ as $\hat{\alpha}_8(s) = \alpha_1^{-1}((\alpha_2(s) + \alpha_3(s)) e^{2(D(0)+\delta_1(s))})$. □

Lemma 13. *There exists a class $\mathcal{K}$ function $\hat{\gamma}^*$ such that for all solutions of the system satisfying (84) for $0 < c < 1$, the following holds*:

$$|\Pi(\theta)| \le \hat{\gamma}^*\left(|X(t)| + \sup_{\phi(t) \le \tau \le t} |Z(\tau)|\right), \quad \phi(t) \le \theta \le t. \tag{130}$$

Proof. Using the change of variable $\theta = \phi(y)$ and (94), we have that

$$\frac{d\Pi(\phi(y))}{dy} = f(\Pi(\phi(y)), \mu(\Pi(\phi(y))) + Z(\phi(y))), \quad t \le y \le \sigma(t). \tag{131}$$

Since $\Pi(\phi(y))$ satisfies the same ODE in y as the ODE for $Y(s)$ in s given by the relation $dY(s)/ds = f(Y(s), \mu(Y(s)) + \omega(s))$, it follows from (103) that

$$|\Pi(\phi(y))| \le \hat{\beta}_2(|X(t)|, y-t) + \hat{\gamma}_1\left(\sup_{t \le y \le \sigma(t)} |Z(\phi(y))|\right)$$

$$\text{for all } t \le y \le \sigma(t). \tag{132}$$

With the fact that $\hat{\beta}(s,r) \le \hat{\beta}(s,0)$ for all $r \ge 0$, we get from (132)

$$|\Pi(\theta)| \le \hat{\beta}_2(|X(t)|,0) + \hat{\gamma}_1\left(\sup_{\phi(t)\le\tau\le t}|Z(\tau)|\right), \quad \phi(t)\le\theta\le t. \tag{133}$$

With the properties of class $\mathcal{K}$ functions we get (130), where $\gamma(s) = \hat{\beta}_2(s,0) + \hat{\gamma}_1(s)$. □

Lemma 14. *There exist class $\mathcal{K}_\infty$ functions $\hat{\alpha}_9$, $\hat{\alpha}_{10}$ such that for all solutions of the system satisfying (84) for $0<c<1$, the following hold*:

$$\hat{\Omega}(t) \le \hat{\alpha}_9(\hat{\Xi}(t)), \tag{134}$$

$$\hat{\Xi}(t) \le \hat{\alpha}_{10}(\hat{\Omega}(t)), \tag{135}$$

for all $t\ge0$, where $\hat{\Omega}$ is defined in (86) and $\hat{\Xi}$ is defined in (96).

Proof. Using the direct backstepping transformation (90) and bounds (126), (118) we get the bound (135) with $\hat{\alpha}_{10}(s) = s + \hat{\alpha}_{11}(\hat{\alpha}_8(s))$. Using the inverse backstepping transformation (93) and the bounds (130), (118) we get the bound (134) with $\hat{\alpha}_9(s) = s + \hat{\alpha}_{11}(\hat{\gamma}^*(s))$. □

Lemma 15. *There exists a function $\hat{\delta}$ of class $\mathcal{K}_\infty$ such that for all solutions of the system that satisfy*

$$\hat{\Omega}(t) < \hat{\delta}^{-1}(c) \quad \text{for all } t\ge0 \tag{136}$$

for $0<c<1$, they also satisfy (84).

Proof. Using (58), (118) one can conclude that if a solution satisfies for all $t\ge0$

$$(|D'(0)| + \delta_2(|\zeta(\theta)|))(|U(\theta)| + (|\nabla\mu(0)| + \hat{\alpha}_{11}(|P(\theta)|))\hat{\alpha}_{11}^*(|P(\theta)| + |\zeta(\theta)|)) < c, \quad \phi(t)\le\theta\le t \tag{137}$$

for $0<c<1$, then it also satisfies (84). Using Lemma 12, (137) is satisfied for $0<c<1$ as long as the bound (136) holds, where the class $\mathcal{K}_\infty$ function $\hat{\delta}$ is given by

$$\hat{\delta}(s) = (|D'(0)| + \delta_2(s))(s + (|\nabla\mu(0)| + \hat{\alpha}_{11}(\hat{\alpha}_8(s)))\hat{\alpha}_{11}^*(\hat{\alpha}_8(s)+s)). \quad \square \tag{138}$$

Lemma 16. *There exists a class $\mathcal{K}$ function ξ_{RoA} such that for all initial conditions of the closed-loop system (76)–(78), (83), (81), (82) that satisfy (85), the solutions of the system satisfy (136) for $0<c<1$ and hence satisfy (84).*

Proof. Using Lemma 14, with the help of (95) we have that

$$\hat{\Omega}(t) \le \hat{\alpha}_9\left(\left(1+\frac{1}{1-c}\right)(\hat{\beta}^*(\hat{\alpha}_{10}(\hat{\Omega}(0)),t) + \hat{\beta}_4(\hat{\alpha}_{10}(\hat{\Omega}(0)), \max\{\{0, t-\sigma(0)\}\}))\right) \tag{139}$$

Hence, for all initial conditions that satisfy the bound (85) with any choice of a class $\mathcal{K}$ function $\xi_{\mathrm{RoA}}(c) \le \overline{\xi}^*_{\mathrm{RoA}}(\hat{\delta}^{-1}(c),c)$, where $\overline{\xi}^*_{\mathrm{RoA}}(s,c)$ is the inverse of the class $\mathcal{KC}_\infty$ function

$$\xi^*_{\mathrm{RoA}}(s,c) = \hat{\alpha}_9\left(\left(1+\frac{1}{1-c}\right)(\hat{\beta}^*(\hat{\alpha}_{10}(s),0) + \hat{\beta}_4(\hat{\alpha}_{10}(s),0))\right), \tag{140}$$

with respect to ξ^*_{RoA}'s first argument, the solutions satisfy (136). Moreover, for all those initial conditions, the solutions verify (84) for all $\theta \ge \phi(0)$. □

Proof of Theorem 2. Using (33), (85) and $0<c<1$ we conclude that $\sigma(0) = D(\zeta(0)) \le D(0) + \delta_1(\xi_{\mathrm{RoA}}(1)) = \xi^*$. Hence, using Corollary 10 in [27] and relation (139) we get (87) with some class $\mathcal{K}_\infty$ function σ_1 where $\hat{\beta}(s,t) = \sigma_1(\hat{\beta}^*(\hat{\alpha}_{10}(s),t) + \hat{\beta}_4(\hat{\alpha}_{10}(s), \max\{\{0, t-\xi^*\}\}))$. Using relations (83), (94) and that fact that $P=\Pi$ we get for all $t\ge0$ that

$$\frac{d\Pi(t)}{dt} = \frac{(1-D'(\mu(\Pi(t))+Z(t))c_Z Z(t)f(\Pi(t),\mu(\Pi(t))+Z(t))}{1-\nabla\mu(\Pi(t))f(\Pi(t),\mu(\Pi(t))+Z(t)))D'(\mu(\Pi(t))+Z(t))}. \tag{141}$$

Under Assumption 1 (Lipschitzness of D'), Assumption 4 (Lipschitzness of $\nabla\mu$) and relation (92) we conclude that the right-hand side of (92), (141) is Lipschitz with respect to (Z,Π) and hence, using also bound (130) there exists a unique solution $(Z(t),\Pi(t)) \in C^1(0,\infty)$. Using (93) we get the existence and uniqueness of $\zeta(t) \in C^1(0,\infty)$. The boundedness of U and (77) guarantee that ζ is continuous at $t=0$. By integrating (77) between any two time instants it is shown that ζ is Lipschitz on $[0,\infty)$ with a Lipschitz constant given by a uniform bound on U. With the fact that $\Pi = P$, relations (83), (84) and the Lipschitzness of D' and $\nabla\mu$ we get the existence and uniqueness of $U \in (0,\infty)$ and that U is locally Lipschitz in $(0,\infty)$. From (76) and (82) we have for all $t>\sigma(0)$ that

$$\dot{X}(t) = f(X(t),\zeta(\phi(t))) \tag{142}$$

$$\dot{\phi}(t) = \frac{1}{1+D'(\zeta(\phi(t)))U(\phi(t))}. \tag{143}$$

Since ζ is Lipschitz on $[0,\infty)$, U is Lipschitz on $(0,\infty)$ and D' is locally Lipschitz, one can conclude that the right hand-side of system (142)–(143) is Lipschitz with respect to (X,ϕ), and hence, there exists a unique solution $(X(t),\phi(t)) \in C^1(\sigma(0),\infty)$. Similarly, the Lipschitzness of the initial conditions $\zeta(s)$ and $U(s)$ for $\phi(0)\le s<0$ guarantees the existence and uniqueness of $(X(t),\phi(t)) \in C^1[0,\sigma(0))$. The boundness of the right-hand side of (142)–(143) guarantees that (X,ϕ) are continuous at $\sigma(0)$, and hence, the Lipschitzness of ζ at 0 guarantees that the right-hand side of (142) is continuous at $\sigma(0)$. Therefore X is continuously differentiable also at $\sigma(0)$. With (84) we get bound (89) and with (33), (136) we get (88) with any class $\mathcal{K}$ function $\hat{\delta}^*(c) \ge \delta_1(\hat{\delta}^{-1}(c))$. □

6. Conclusions

We present a methodology for the compensation of state-dependent delays that depend on delayed states, by designing a predictor feedback law. We prove asymptotic stability of the closed-loop system with the aid of a Lyapunov functional that we construct by introducing a backstepping transformation. The results of this paper can be directly extended to the case in which the delay is explicitly defined as a function of past values of the state, at a time instant that is a priori given. The present results seem also extendable to the case in which the delay might depend on past values of the state, but at a time instant that may be a function of the delay (rather than just identical to the delay).

Since in this paper we deal with delays that depend on delayed states, whereas in [3] we deal with delays that depends on current states, it is reasonable to ask whether this methodology can be extended to the case in which the delay function depends both on delayed and current states. For designing the predictor feedback law for such a delay function one has first to show the well-posedness of both the prediction and delay times (namely σ and ϕ respectively). To show this one has to study the existence and uniqueness of a two-point boundary value problem for the prediction and the delay times. For studying the existence and uniqueness of this problem one has to use fixed-point theory incorporating the properties of the dynamics of these two times and the properties of the solutions of the system. This study is far from trivial and can be pursued in the future.

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