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Hybrid boundary stabilization of linear first-order hyperbolic PDEs despite almost quantized measurements and control input

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A B S T R A C T

We develop a hybrid boundary feedback law for a class of scalar, linear, first-order hyperbolic PDEs, for which the state measurements or the control input are subject to quantization. The quantizers considered are Lipschitz functions, which can approximate arbitrarily closely typical piecewise constant, taking finitely many values, quantizers. The control design procedure relies on the combination of two ingredients—A nominal backstepping controller, for stabilization of the PDE system in the absence of quantization, and a switching strategy, which updates the parameters of the quantizer, for compensation of the quantization effect. Global asymptotic stability of the closed-loop system is established through utilization of Lyapunov-like arguments and derivation of solutions' estimates, providing explicit estimates for the supremum norm of the PDE state, capitalizing on the relation of the resulting, nonlinear PDE system (in closed loop) to a certain, integral delay equation. A numerical example is also provided to illustrate, in simulation, the effectiveness of the developed design.

1. Introduction

Although for boundary controlled, linear, first-order hyperbolic PDE systems delay, see, for example, [1–5]; sampling, see, for instance, [6–9]; and saturation, see, e.g., [10], effects are addressed in existing control designs[1] and despite the results [15,16], in which practical stabilization is achieved (as the considered quantizers are static and the control design approaches do not, explicitly, aim at treating state measurements errors due to quantization), the problem of compensation of quantization in measurements and control input, achieving global asymptotic stabilization, has not been, heretofore, investigated. Besides its theoretical significance, the actual motivation behind studying this problem is attributed to the aim of addressing yet another important practical issue embedded in actual, feedback control loops involving PDE systems. In this paper we launch an effort in this direction considering a specific class of boundary controlled, scalar, first-order hyperbolic PDEs. We aim at providing a first step toward systematic treatment of quantization effects in other (potentially, more general) classes of PDE systems, while keeping the technical burden at a level that does not obscure the key design and analysis ideas of the developed approach.

We construct a hybrid feedback law, which is based on combination of two elements—A nominal backstepping control design [17], which achieves stabilization of the PDE system in the absence of quantization, and a switching update law for the tunable parameter of the quantizer [18], which achieves compensation of the quantization effect. The PDE state in the backstepping controller is replaced by its quantized form, while the so-called "zoom" variable of the dynamic quantizer is updated at discrete time instants. In particular, the switching strategy consists of two phases. During the "zooming out" stage, the adjustable parameter of the quantizer is increasing, in a piecewise constant manner, until a certain event is detected, using only quantized measurements of the PDE state. From that time instant on is decreasing, in a piecewise constant fashion, in which switches occur at a priori specified, equidistant time instants,[2] which depend on the plant parameters, through the backstepping kernels, as well as the fixed parameters of the quantizer.

Global asymptotic stability of the closed-loop system, in the supremum norm of the PDE state, is established, capitalizing on the relation of first-order hyperbolic PDEs to integral delay equations [19]. Within the presented framework, in which we treat quantizers in correspondence to the case of finite-dimensional systems, the choice of the supremum norm is necessary. The reason is that, in both the control design and stability analysis performed, it is crucial to relate the magnitude of the PDE state to the magnitude of its quantized version and vice versa. This is enabled (without involving higher, spatial derivatives of the PDE state)

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1 See also, e.g., [11–14] for respective results in the parabolic case.

2 Event-based switching, on the basis of the available quantized measurements, is also possible. Nevertheless, such an update law for the adjustable parameter of the quantizer is not studied here.

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considering the supremum norm of the PDE state. The stability analysis is performed dividing the time horizon into two different intervals. Within the first, constructing a proper estimate on open-loop solutions, it is shown that their exists a time instant (the upper bound of which depends on the plant/quantizer parameters and size of initial conditions) at which the magnitude of the quantizer's adjustable parameter is sufficiently larger than the size of the state, in the sense of its supremum norm (in other words, the PDE state of the system can be adequately measured as it is within the range of the quantizer). Within the second, for each time period in which the zoom variable of the quantizer remains constant, with the aid of a proper Lyapunov-like functional that is constructed, it is shown that the solutions of the closed-loop system are ultimately bounded, in the sense of their supremum norm, with a bound that is proportional to the magnitude of the adjustable parameter of the quantizer. As the zoom variable, and thus, also the obtained ultimate bounds of the solutions, decreases, global asymptotic stability can be established. In particular, explicit stability estimates in the supremum norm of the PDE state are derived, which exhibit an exponential decay rate (for equidistant switching instants) of the norm of the solutions, with an overshoot coefficient that depends on a certain power (in turn, dependent on the plant/quantizer parameters) of the norm of initial conditions.

The quantizers considered in the present paper differ from respective quantizers considered in, e.g., [18,20], in that it is assumed they are locally Lipschitz functions, rather than only piecewise constant, taking values in a finite subset of the real numbers. The reason for imposing such an assumption is to guaranteeing the well-posedness of the closed-loop system, which is established employing the results in [19] (thus without needing to study the issue of existence and uniqueness of solutions in full generality, which would be out of the scope of this paper that focuses on the control design and stability analysis). Although this technical condition may appear as a restrictive requirement, in practice, it is not. This is illustrated presenting an example of an approximate quantizer, which may be viewed as a typical quantizer with rectilinear quantization regions, considered in, e.g., [18,20], with an ε-layer added around the points of discontinuity, which could be taken as arbitrarily small. In fact, the derived stability estimates would be independent of the size of such a layer, which suggests that they may remain valid even in the absence of the layer (this is demonstrated in the simulation example, in which the quantizer is chosen only as piecewise constant, taking finitely many values).

Although the central design and analysis ideas are similar to the case of state quantization, we also develop a respective hybrid feedback law for the case in which state measurements are available, yet, the control input signal is subject to quantization. In such a case, the hybrid feedback law is expressed as the quantized form of the nominal backstepping controller, while the adjustable parameter of the quantizer is chosen in a way analogous to the case of state measurements quantization. Closed-loop stability can be established in a quite similar manner to the case of state quantization.

In Section 2 we present the class of systems under investigation and the developed hybrid feedback law. Section 3 incorporates the main result of the paper, namely, establishment of closed-loop stability, under the proposed control law, despite quantized measurements. Section 4 presents the extension of our developments to the case of input quantization. Section 5 provides a numerical example and in Section 6 we discuss potential, future research extensions.

Notation. We denote by $L^\infty(A;\Omega)$ the space of measurable and bounded functions defined on A and taking values in Ω. For a given $D>0$ and a function $u\in L^\infty([0,D];\mathbb{R})$ we define $\|u\|_\infty=\sup_{x\in[0,D]}|u(x)|$. For $\eta_t\in L^\infty([-D,0];\mathbb{R}^n)$, the notation η_t refers to $\eta_t(s)=\eta(t+s)$, for $s\in[-D,0]$. We define $\|\eta_t\|=\sup_{-D\le s\le 0}|\eta(t+s)|$. For a given $h\in\mathbb{R}$ we define its integer part as $\lfloor h\rfloor=\max\{k\in\mathbb{Z}:k\le h\}$.

2. Problem formulation and control design

2.1. Linear first-order hyperbolic PDEs in strict-feedback form

We consider the following system

$$u_t(x,t)=u_x(x,t)+g(x)u(0,t)+\int_0^x\bar g(x,y)u(y,t)dy \tag{1}$$

$$u(D,t)=U(t), \tag{2}$$

where $x\in[0,D]$, with $D>0$, is spatial variable, $t\ge 0$ is time variable, u is scalar state, and U is control input, with g and $\bar g$ continuous functions. System (1), (2) can be transformed to

$$w_t(x,t)=w_x(x,t) \tag{3}$$

$$w(D,t)=U(t)-\int_0^D k(D,x)\left(w(x,t)+\int_0^x l(x,y)w(y,t)dy\right)dx, \tag{4}$$

employing the backstepping transformation and its inverse, provided in [17] as

$$w(x,t)=u(x,t)-\int_0^x k(x,y)u(y,t)dy \tag{5}$$

$$u(x,t)=w(x,t)+\int_0^x l(x,y)w(y,t)dy, \tag{6}$$

satisfying

$$M_2\|u\|_\infty\le\|w\|_\infty\le M_1\|u\|_\infty, \tag{7}$$

where M_1, M_2 are defined through the continuously differentiable kernels k and l as

$$M_1=1+D\max_{0\le x\le D}\max_{0\le y\le x}|k(x,y)| \tag{8}$$

$$M_2=\frac{1}{1+D\max_{0\le x\le D}\max_{0\le y\le x}|l(x,y)|}. \tag{9}$$

Although the procedure involving (3)–(9) is known, we display these equations as the hybrid control design relies on the backstepping transformation and depends explicitly on parameters (8), (9).

There is no conceptual obstacle to extending the presented results to the case of a more general class of scalar, linear, first-order hyperbolic PDE systems than (1), (2), as long as there exists an invertible, integral transformation that maps the original system to the transport PDE system (3) with boundary condition of the form (4). Such an example could be the class of systems considered in [21], which incorporates non-strict-feedback (and reaction) terms on the right-hand side of (1), (2), employing Fredholm-type integral transformations. However, we sacrifice generality, in order to not distract a reader, involving additional technical details primarily related to existing results, from the main purpose of the paper, which is presentation of the hybrid control design methodology.

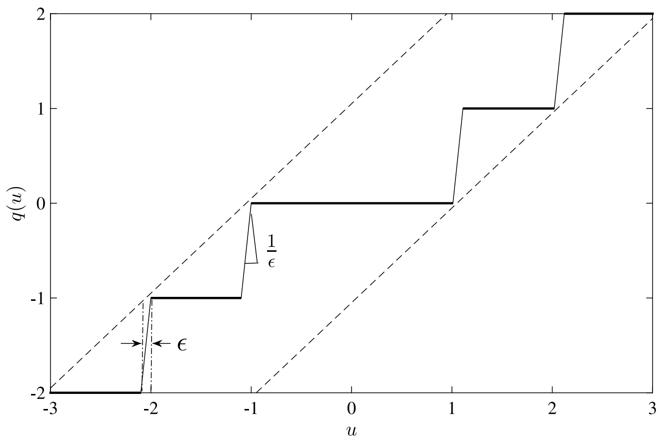

Fig. 1. The approximate quantizer with ε-layer defined in (11) (where $\varepsilon = 0.1$).

2.2. Approximate quantizer with an adjustable parameter

The state u is available only in quantized form, in which the quantizer q_μ is defined as (see, e.g., [18])

$$q_\mu(u) = \mu q\left(\frac{u}{\mu}\right), \tag{10}$$

where $\mu > 0$ can be manipulated and q is a locally Lipschitz function that satisfies the following properties

P1: If $|u| \le M$, then $|q(u) - u| \le \Delta$.
P2: If $|u| > M$, then $|q(u)| > M - \Delta$.
P3: If $|u| \le \bar{M}$, then $q(u) = 0$,

for some positive constants M, $\bar{M}$, and Δ, with $M > \Delta$. In particular, the fixed parameters M and Δ are referred to as the range and quantization error of the quantizer, respectively, whereas the adjustable parameter μ is referred to as the "zoom" variable.

An example of a quantizer that may arbitrarily closely approximate a typical quantizer with rectilinear quantization regions (see, e.g., [20]) is given below and shown in Fig. 1

$$q(u) = \begin{cases} 2, & u \ge 3 \\ -2, & u \le -3 \\ \frac{u-k}{\varepsilon} + k - 1, & k \le u \le k + \varepsilon \\ k, & k + \varepsilon \le u \le k + 1 \\ -k, & -(k+1) \le u \le -k - \varepsilon \\ \frac{u+k}{\varepsilon} - k + 1, & -k - \varepsilon \le u \le -k \\ 0, & -1 \le u \le 1 \end{cases}, \tag{11}$$

where $k = 1, 2$. Quantizer (11) satisfies properties P1–P3 (with $M = 3$, $\bar{M} = 1$, and $\Delta = 1.05$), while being Lipschitz. Given the parameters M and Δ of the quantizer, the derived stability estimates, presented in the next two sections, for the closed-loop system, do not depend explicitly on the value of $\varepsilon > 0$, and thus, they hold for an arbitrarily small $\varepsilon > 0$. For this reason, one could even argue that the stability analysis would remain valid for $\varepsilon = 0$. This is demonstrated in Section 5, in which we provide a specific example of a quantizer, which satisfies properties P1–P3, yet, it is only piecewise constant, taking finitely many values. For further examples and details regarding quantizers the reader is referred to, e.g., [18,20].

2.3. Boundary hybrid feedback law using quantized measurements

The hybrid feedback law is based on the quantized version of the backstepping controller and a suitably chosen piecewise constant signal μ. It is defined as

$$U(t) = \begin{cases} 0, & 0 \le t \le t_1^* \\ \mu(t) \int_0^D k(D, x)\, q\left(\frac{u(x,t)}{\mu(t)}\right) dx, & t > t_1^* \end{cases}, \tag{12}$$

where for some fixed, yet arbitrary, $\tau, \mu_0 > 0$

$$\mu(t) = \begin{cases} \max\left\{1, D\bar{M}_1\right\} e^{2\bar{M}_1 j\tau} \mu_0, & (j-1)\tau \le t \le j\tau, \quad 1 \le j \le \left\lfloor \frac{t_1^*}{\tau} \right\rfloor + 1 \\ \mu\left(t_1^*\right), & t_1^* < t \le t_1^* + T \\ \Omega \mu\left(t_1^* + (i-1)T\right), & t_1^* + (i-1)T < t \le t_1^* + iT, \quad i = 2, \ldots \end{cases}, \tag{13}$$

with t_1^* being the first time instant at which the following holds

$$\left|q_\mu\left(u\left(x, t_1^*\right)\right)\right| \le \left(\frac{M}{(1+\lambda)^\nu} \frac{M_2}{M_1} - \Delta\right) \mu\left(t_1^*\right), \quad \text{for all } x \in [0, D], \tag{14}$$

where $0 < \nu < 1$ and $\lambda > 0$ are fixed, yet arbitrary, parameters, and

$$\bar{M}_1 = \frac{M_3}{DM_2} \tag{15}$$

$$M_3 = D \sup_{0 \le y \le D} |k(D, y)| \tag{16}$$

$$\Omega = \frac{(1+\lambda)^{1+\nu} \Delta M_3}{M_2 M} \tag{17}$$

$$T = -D \frac{\ln \Omega}{\nu \ln(1+\lambda)}. \tag{18}$$

Event (14) can be detected using measurements of $q_\mu(u(x))$ and μ only.[3] Furthermore, a, potentially, more practical way to detect this event would be to alternatively check whether relation $\left\|q_\mu\left(u\left(t_1^*\right)\right)\right\|_\infty \le \left(\frac{M}{(1+\lambda)^\nu} \frac{M_2}{M_1} - \Delta\right) \mu\left(t_1^*\right)$ holds, which guarantees that (14) also holds.

The switching strategy (13) involves, via Ω and T, the fixed parameters of the quantizer and the plant parameters (through the backstepping kernels). The parameters ν and λ could be tuned. From the explicit stability estimate for the closed-loop system presented immediately next, it is evident that increasing ν and λ guarantees faster decay rate, nevertheless, at the expense of increase in the initial overshoot. Furthermore, relations (17), (18) also imply that increasing ν and λ results in faster switching (due to decrease of T). The choice of the parameter τ is guided only by the desirable switching frequency during the initial time interval, in which the system operates in open loop. Parameter μ_0 could be chosen in order to scale the overshoot coefficient involved in the response of the system as the increase in the value of μ_0 results, in general, in a decrease in the value of t_1^*.

3. Stability analysis of the hybrid backstepping controller under state quantization

Theorem 1. *Consider the closed-loop system consisting of the plant (1), (2) and the hybrid feedback law (12), (13) with parameters (15)–(18). If Δ and M satisfy $\frac{\Delta}{M} < \frac{M_2}{M_1(1+\lambda)^\nu \max\{1+\lambda, 2\}}$, then for all $u(\cdot, 0) \in L^\infty([0, D]; \mathbb{R})$ there exists a unique solution $u(\cdot, t) \in L^\infty([0, D]; \mathbb{R})$, which satisfies*

$$\|u(t)\|_\infty \le \gamma \|u(0)\|_\infty^{2 + \frac{\nu \ln(1+\lambda)}{D\bar{M}_1}} e^{-\frac{\nu \ln(1+\lambda)}{D} t}, \quad t \ge 0, \tag{19}$$

[3] The requirement of (14) to hold for all x should be understood as a requirement for all x for which the solution exists.

where

$$\gamma = \frac{\max\left\{1, D\bar{M}_1\right\}}{M_2} \max\left\{\frac{M_2 M}{\Omega} e^{2\bar{M}_1 \tau} \mu_0, M_1\right\} \max \times \left\{\frac{M_1}{\mu_0 M_2\left(\frac{M}{(1+\lambda)^\nu}\frac{M_2}{M_1} - 2\Delta\right)}, 1\right\} \times \left(\frac{M_1}{\mu_0 M_2\left(\frac{M}{(1+\lambda)^\nu}\frac{M_2}{M_1} - 2\Delta\right)}\right)^{1+\frac{\nu \ln(1+\lambda)}{D\bar{M}_1}}. \tag{20}$$

For given data of the plant, i.e., for given parameters M_1, M_2, and M_3 (which depend on the plant parameters through the kernels of the backstepping transformation and its inverse, according to definitions (8), (9), and (16), respectively), in order for the condition of Theorem 1 to be satisfied the parameters of the quantizer Δ and M should be such that Δ is sufficiently smaller than M. We note that this condition also implies $\frac{\Delta M_3}{M_2 M} < \frac{1}{(1+\lambda)^{1+\nu}}$, as $M_3 < M_1$, which guarantees that $\Omega < 1$. In particular, as long as the parameters of the plant and quantizer satisfy the condition $\frac{\Delta M_1}{M M_2} < \frac{1}{2(1+c)}$, for some $c > 0$, there always exist $0 < \nu < 1$ and $\lambda > 0$ such that the condition of Theorem 1 holds. The conservatism of this condition may be interpreted qualitatively from the viewpoint of the degree of restriction of the allowable range of the quantizer parameters, for given data of the plant. For instance, an increase in the magnitude of g (potentially considering a more unstable plant), results in a decrease on the allowable range of $\frac{\Delta}{M}$ as M_1 increases and M_2 decreases.

The specific power involved in the supremum norm of the initial state in estimate (19) arises from derivation of an upper bound on open-loop solutions of exponential type together with the derivation of an upper bound for time instant t_1^*, which depends on a logarithmic function of the magnitude of the initial state. The particular decay rate obtained for the supremum norm of the solutions arises from the equidistant switching instants (within time interval $t > t_1^*$) and the fact that μ is decreasing by a factor of Ω at each switching instant.

To avoid incorporating, unnecessarily, additional notational burden in the statement of Theorem 1, it is tacitly assumed that $t_1^* > 0$, which represents the most difficult and generic case. In the occasion where $t_1^* = 0$, estimate (19) could be replaced by the simplest expression $\|u(t)\|_\infty \le \max\left\{\frac{\mu_0 \max\{1, D\bar{M}_1\} M e^{2\bar{M}_1\tau}}{\Omega}, \|u(0)\|_\infty\right\} e^{-\frac{\nu \ln(1+\lambda)}{D}t}$, which further implies from (17) and the condition on $\frac{\Delta}{M}$ within the statement of Theorem 1 that $\|u(t)\|_\infty \le \frac{\mu_0 \max\{1, D\bar{M}_1\} M e^{2\bar{M}_1\tau}}{\Omega} e^{-\frac{\nu \ln(1+\lambda)}{D}t}$, as the initial state satisfies relation (21).

The proof of Theorem 1 is based on the following two lemmas and capitalizes on the relation of the first-order hyperbolic PDE (3), (4), under (12), to a specific integral delay equation [19] (see also, e.g., [22–24]). The first lemma deals with the so-called "zooming out" phase [18], in which it is established that the state is within the range of the quantizer at time t_1^*, whereas the second lemma deals with the "zooming-in" phase, establishing, for fixed μ, a certain ultimate boundedness property, which in turn is employed for piecewise constant μ, in each time interval of the form $\left(t_1^* + (i-1)T, t_1^* + iT\right]$, $i = 1, 2, \ldots$, for showing asymptotic stability of the closed-loop system.

Lemma 1. *Under the constraint for Δ and M stated in Theorem 1, there exists a time instant t_1^*, satisfying $t_1^* \le \frac{1}{\bar{M}_1} \ln \frac{\frac{M_1}{\mu_0}\|u(0)\|_\infty}{M_2\left(\frac{M}{(1+\lambda)^\nu}\frac{M_2}{M_1} - 2\Delta\right)}$, such that (14) holds, and hence, the following also holds*

$$\left|u\left(x, t_1^*\right)\right| \le \frac{M}{(1+\lambda)^\nu}\frac{M_2}{M_1}\mu\left(t_1^*\right), \quad \text{for all } x \in [0, D]. \tag{21}$$

Proof. Using the equivalent representation of the transformed system (3), (4) through an integral delay equation and setting $w(x,t) = \eta(t + x - D)$, we obtain with (12) the following integral delay equation for $0 < t \le t_1^*$

$$\eta(t) = -\int_{-D}^{0}\left(k(D, s+D) + \int_s^0 k(D, r+D) l(r+D, s+D) dr\right) \times \eta(t+s) ds. \tag{22}$$

Using the left-hand side of (7) together with (15) we get from (22) that

$$|\eta(t)| \le \bar{M}_1 \int_{t-D}^{t} |\eta(s)|\, ds. \tag{23}$$

Hence, we obtain the following two relations

$$|\eta(t)| \le \bar{M}_1\left(\int_{-D}^{0}|\eta(s)|\, ds + \int_0^t |\eta(s)|\, ds\right), \quad 0 < t \le D, \tag{24}$$

and

$$|\eta(t)| \le \bar{M}_1\left(\int_{0}^{D}|\eta(s)|\, ds + \int_D^t |\eta(s)|\, ds\right), \quad t > D. \tag{25}$$

Relation (24) implies (see, e.g., [25]) that

$$|\eta(t)| \le D e^{\bar{M}_1 t}\bar{M}_1\|\eta_0\|, \quad 0 < t \le D, \tag{26}$$

and hence, we obtain using (25)

$$|\eta(t)| \le D\bar{M}_1\left(e^{\bar{M}_1 D} - 1\right)\|\eta_0\| e^{\bar{M}_1(t-D)}, \quad t > D. \tag{27}$$

Therefore, using the fact that $\sup_{-D \le s \le 0}|\eta(t+s)| \le \max\left\{\|\eta_0\|, \sup_{0<s\le t}|\eta(s)|\right\}$, for $0 < t \le D$, we get from (26) that

$$\|\eta_t\| \le \max\left\{1, D\bar{M}_1\right\}\|\eta_0\| e^{\bar{M}_1 t}, \quad 0 \le t \le D. \tag{28}$$

Similarly, using the fact that $\sup_{-D \le s \le 0}|\eta(t+s)| \le \max\left\{\sup_{0 \le s \le D}|\eta(s)|, \sup_{D<s\le t}|\eta(s)|\right\}$, for $t > D$, we obtain from (26), (27)

$$\|\eta_t\| \le D\bar{M}_1\|\eta_0\| e^{\bar{M}_1 t}, \quad t > D. \tag{29}$$

Hence, combining (28), (29) we get

$$\|\eta_t\| \le \max\left\{1, D\bar{M}_1\right\}\|\eta_0\| e^{\bar{M}_1 t}, \quad 0 \le t \le t_1^*, \tag{30}$$

which also implies that

$$\|w(t)\|_\infty \le \max\left\{1, D\bar{M}_1\right\} e^{\bar{M}_1 t}\|w(0)\|_\infty, \quad 0 \le t \le t_1^*. \tag{31}$$

In the above analysis we tacitly assume that $t_1^* > D$. However, estimate (31) still holds in the case where $t_1^* \le D$ as one could directly derive it considering only the case $0 \le t \le t_1^* \le D$. Choosing the switching signal μ according to (13) one can conclude that there exists a time t_1^*, which is at most equal to $\frac{1}{\bar{M}_1} \ln \frac{\frac{\|w(0)\|_\infty}{\mu_0}}{M_2\left(\frac{M}{(1+\lambda)^\nu}\frac{M_2}{M_1} - 2\Delta\right)}$, such that

$$\frac{\|w\left(t_1^*\right)\|_\infty}{\mu\left(t_1^*\right)} \le M_2\left(\frac{M}{(1+\lambda)^\nu}\frac{M_2}{M_1} - 2\Delta\right), \tag{32}$$

and hence, in view of property P1 of the quantizer and the fact that for all $0 \le x \le D$ it holds that $\frac{|u(x,t_1^*)|}{\mu(t_1^*)} \le \frac{\|u(t_1^*)\|_\infty}{\mu(t_1^*)} \le \frac{1}{M_2}\frac{\|w(t_1^*)\|_\infty}{\mu(t_1^*)} \le \frac{1}{(1+\lambda)^\nu} M \frac{M_2}{M_1} - 2\Delta$, where $M_2 \le M_1$, we obtain, using triangular inequality, that the following holds

$$\left|q\left(\frac{u\left(x,t_1^*\right)}{\mu\left(t_1^*\right)}\right)\right| \leq \frac{M}{(1+\lambda)^{\nu}}\frac{M_2}{M_1}-\Delta,\quad \text{for all } x\in[0,D]. \tag{33}$$

Equivalently, at time t_1^* relation (14) holds. We then show that detecting the event (14) in combination with the properties of the quantizer implies that at t_1^* relation (21) also holds. Given that (33) holds, we first check whether there could be some $x \in [0, D]$, say x_1^*, such that relation $\frac{|u(x_1^*,t_1^*)|}{\mu(t_1^*)} > M$ holds. If that was the case, then from property P2 of the quantizer one gets that relation $\left|q\left(\frac{u(x_1^*,t_1^*)}{\mu(t_1^*)}\right)\right| > M-\Delta$ would hold. However, since $M_2 \leq M_1$, one can conclude that relation $\left|q\left(\frac{u(x_1^*,t_1^*)}{\mu(t_1^*)}\right)\right| > \frac{M_2}{M_1}\frac{M}{(1+\lambda)^{\nu}}-\Delta$ would also hold, which contradicts (33), and hence, for all $x \in [0, D]$ it holds that $\frac{|u(x,t_1^*)|}{\mu(t_1^*)} \leq M$. We next check whether relation $\frac{M_2}{M_1}\frac{M}{(1+\lambda)^{\nu}} < \frac{|u(x_1^*,t_1^*)|}{\mu(t_1^*)} \leq M$ could hold. In that case, property P1 of the quantizer would imply that the following would hold

$$\left|q\left(\frac{u\left(x_1^*,t_1^*\right)}{\mu\left(t_1^*\right)}\right)-\frac{u\left(x_1^*,t_1^*\right)}{\mu\left(t_1^*\right)}\right| \leq \Delta, \tag{34}$$

and hence, using the fact that $\left|\frac{u(x_1^*,t_1^*)}{\mu(t_1^*)}\right| - \left|q\left(\frac{u(x_1^*,t_1^*)}{\mu(t_1^*)}\right)\right| \leq \left|q\left(\frac{u(x_1^*,t_1^*)}{\mu(t_1^*)}\right)-\frac{u(x_1^*,t_1^*)}{\mu(t_1^*)}\right|$, which follows from triangular inequality, we arrive at

$$\left|\frac{u\left(x_1^*,t_1^*\right)}{\mu\left(t_1^*\right)}\right| - \left|q\left(\frac{u\left(x_1^*,t_1^*\right)}{\mu\left(t_1^*\right)}\right)\right| \leq \Delta. \tag{35}$$

Relation (35) in combination with $\frac{M_2}{M_1}\frac{M}{(1+\lambda)^{\nu}} < \frac{|u(x_1^*,t_1^*)|}{\mu(t_1^*)}$ would imply that

$$\frac{M_2}{M_1}\frac{M}{(1+\lambda)^{\nu}}-\Delta < \left|q\left(\frac{u\left(x_1^*,t_1^*\right)}{\mu\left(t_1^*\right)}\right)\right|, \tag{36}$$

which contradicts (33), establishing the validity of (21). □

Lemma 2. *Under the constraint for Δ and M stated in Theorem 1, the solutions of the transformed closed-loop system (3), (4), (12), for fixed $\mu > 0$, which satisfy*

$$\sup_{0\leq x\leq D}\left|e^{(x-D)\frac{\nu\ln(1+\lambda)}{D}}w\left(x,t_1^*\right)\right| \leq \frac{1}{(1+\lambda)^{\nu}}M_2M\mu, \tag{37}$$

they also satisfy for $t_1^* < t < t_1^*+T$

$$\sup_{0\leq x\leq D}\left|e^{(x-D)\frac{\nu\ln(1+\lambda)}{D}}w(x,t)\right| \leq \max\left\{\sup_{0\leq x\leq D}\left|e^{(x-D)\frac{\nu\ln(1+\lambda)}{D}}w\left(x,t_1^*\right)\right|,\ \frac{\Omega}{(1+\lambda)^{\nu}}M_2M\mu\right\}. \tag{38}$$

Moreover, the following holds

$$\sup_{0\leq x\leq D}\left|e^{(x-D)\frac{\nu\ln(1+\lambda)}{D}}w\left(x,t_1^*+T\right)\right| \leq \frac{1}{(1+\lambda)^{\nu}}\Omega M_2M\mu. \tag{39}$$

Proof. Under the feedback law (12) for $t > t_1^*$ (where u is expressed in terms of w via (6)) and since μ is fixed within the interval $t_1^* < t \leq t_1^*+T$ we get the following integral delay equation equivalent to (3), (4)

$$\eta(t)=f\left(\eta_t,\mu\right), \tag{40}$$

where

$$f\left(\eta_t,\mu\right) = \mu\int_{-D}^{0}k\left(D,s+D\right)\left(q\left(\frac{1}{\mu}G(t,s)\right)-\frac{1}{\mu}G(t,s)\right)ds \tag{41}$$

$$G(t,s) = \eta(t+s)+\int_{-D}^{s}l(s+D,r+D)\eta(t+r)dr. \tag{42}$$

Defining $V_1(\eta) = \sup_{-D\leq s\leq 0} e^{s\frac{\nu\ln(1+\lambda)}{D}}|\eta(s)|$ we get for sufficiently small $h > 0$

$$\begin{aligned}V_1\left(\eta_{t+h}\right) &= \sup_{-D\leq s\leq 0} e^{s\frac{\nu\ln(1+\lambda)}{D}}|\eta(t+s+h)| \\ &\leq \max\left\{\sup_{-D\leq s\leq -h} e^{s\frac{\nu\ln(1+\lambda)}{D}}|\eta(t+s+h)|,\right. \\ &\qquad \left.\sup_{-h\leq s\leq 0} e^{s\frac{\nu\ln(1+\lambda)}{D}}|\eta(t+s+h)|\right\} \\ &\leq \max\left\{e^{-h\frac{\nu\ln(1+\lambda)}{D}}\sup_{h-D\leq s\leq 0} e^{s\frac{\nu\ln(1+\lambda)}{D}}|\eta(t+s)|,\right. \\ &\qquad \left.\sup_{0\leq s\leq h}|\eta(t+s)|\right\},\end{aligned} \tag{43}$$

and hence, with definition for V_1 we arrive at

$$V_1\left(\eta_{t+h}\right) \leq \max\left\{e^{-h\frac{\nu\ln(1+\lambda)}{D}}V_1\left(\eta_t\right),\ \sup_{0\leq s\leq h}|\eta(t+s)|\right\}. \tag{44}$$

As long as $\frac{1}{(1+\lambda)^{\nu}}\Omega M_2M\mu \leq \|\eta\| \leq M_2M\mu$, using the property P1 in Section 2.2 of the quantizer and the left-hand side of bound (7) we arrive using (41) at

$$\begin{aligned}|f(\eta,\mu)| &\leq \mu M_3\Delta \\ &\leq \frac{(1+\lambda)^{\nu}M_3\Delta}{\Omega M_2M}\|\eta\|,\end{aligned} \tag{45}$$

and thus, employing definition (17) we get that

$$|f(\eta,\mu)| \leq \frac{1}{1+\lambda}\|\eta\|. \tag{46}$$

Therefore, along the solutions of (40), using (46) we obtain

$$|\eta(t+q)| \leq \frac{1}{1+\lambda}\max\left\{\sup_{-D+q\leq s\leq 0}|\eta(t+s)|,\ \sup_{0\leq s\leq q}|\eta(t+s)|\right\}, \tag{47}$$

for $0 \leq q \leq h$, and hence, since $V_1(\eta) \geq e^{-\nu\ln(1+\lambda)}\sup_{-D\leq s\leq 0}|\eta(s)|$, we get that

$$\sup_{0\leq q\leq h}|\eta(t+q)| \leq \frac{1}{1+\lambda}\max\left\{e^{\nu\ln(1+\lambda)}V_1\left(\eta_t\right),\ \sup_{0\leq s\leq h}|\eta(t+s)|\right\}. \tag{48}$$

Thus,

$$\sup_{0\leq s\leq h}|\eta(t+s)| \leq \frac{1}{1+\lambda}e^{\nu\ln(1+\lambda)}V_1\left(\eta_t\right). \tag{49}$$

Combining (44), (49) we get that for all h such that $0 < h \leq \min\left\{D\left(\frac{1}{\nu}-1\right), T\right\}$ the following holds

$$V_1\left(\eta_{t+h}\right) \leq e^{-h\frac{\nu\ln(1+\lambda)}{D}}V_1\left(\eta_t\right). \tag{50}$$

By induction[4] we obtain for $t_1^* < t \leq t_1^*+T$

$$V_1\left(\eta_t\right) \leq e^{-\frac{\nu\ln(1+\lambda)}{D}\left(t-t_1^*\right)}V_1\left(\eta_{t_1^*}\right), \tag{51}$$

[4] In more detail, we express time as $t = t_1^*+ih+q$, where $0 \leq i \leq \left\lfloor\frac{T}{h}\right\rfloor$ is integer and q is a real number such that $0 \leq q \leq h$ (see, e.g., [19] for detailed computations).

and hence, from the definition of V_1 we get that

$$\|\eta_t\| \leq \mathrm{e}^{\nu \ln(1+\lambda)} \mathrm{e}^{-\frac{\nu \ln(1+\lambda)}{D}(t-t_1^*)} \sup_{-D\leq s\leq 0} \mathrm{e}^{s\frac{\nu \ln(1+\lambda)}{D}} \left|\eta\left(t_1^*+s\right)\right|, \quad (52)$$

which implies from (37) that $\|\eta_t\| \leq M_2 M\mu$, $t_1^* < t \leq t_1^* + T$. Furthermore, using (51), from the definition of V_1 and (18) we get that

$$\sup_{-D\leq s\leq 0} \mathrm{e}^{s\frac{\nu \ln(1+\lambda)}{D}} \left|\eta\left(t_1^*+T+s\right)\right| \leq \Omega \sup_{-D\leq s\leq 0} \mathrm{e}^{s\frac{\nu \ln(1+\lambda)}{D}} \left|\eta\left(t_1^*+s\right)\right|, \quad (53)$$

and hence, employing (37) we obtain (39), which also implies the following ultimate bound

$$\|\eta_{t_1^*+T}\| \leq \Omega M_2 M\mu. \quad (54)$$

The analysis, in fact, provides an upper bound on the time for which relation (39) is satisfied, while it remains valid (and thus, so it remains estimate (51)) as long as the solutions satisfy $\frac{1}{(1+\lambda)^{\nu}}\Omega M_2 M\mu \leq \|\eta_t\|$ (and thus, based on the preceding analysis, they also satisfy $\|\eta_t\| \leq M_2 M\mu$). In the case where the solutions satisfy $\|\eta_{t_1}\| \leq \frac{1}{(1+\lambda)^{\nu}}\Omega M_2 M\mu = (1+\lambda)\Delta M_3\mu$ for some t_1 such that $t_1^* \leq t_1 \leq t_1^* + T$, then they also satisfy $\|\eta_t\| \leq \frac{1}{(1+\lambda)^{\nu}}\Omega M_2 M\mu$, for $t_1 \leq t \leq t_1^* + T$, fact which, in combination with (51) and the fact that $V_1(\eta) \leq \|\eta\|$, establish bound (38). This is shown exploiting the form of the integral delay equation (40) noting from the first equation in (45) that the implication $\|\eta\| \leq (1+\lambda)\Delta M_3\mu \implies |f(\eta,\mu)| \leq \mu M_3\Delta$ holds. □

The Lyapunov-like functional employed within the proof of Lemma 2 for establishing an ultimate boundedness property of the target system, is defined as $V_1(\eta) = \sup_{-D\leq s\leq 0} \mathrm{e}^{s\frac{\nu\ln(1+\lambda)}{D}}|\eta(s)|$, motivated by Theorem 2.6 in [19] (for the case of no input), utilizing the particular integral delay equation representation (40)–(42). However, as in Lemma 2 the goal is to establish an ultimate boundedness property for the target system, rather than asymptotic stability, one cannot directly apply Theorem 2.6 in [19]. Instead, we properly adapt its proof strategy for establishing an ultimate boundedness property, in principle, also inspired by a respective analysis for finite-dimensional systems. Furthermore, as we are not aware of an off-the-shelf, Lyapunov-like theorem that we could employ for directly establishing the required ultimate boundedness property for system (40)–(42) and since it is crucial to provide a constructive way for deriving an estimate of time T (i.e., an upper bound of the time interval needed for the solutions to enter a desired region), we explicitly derive stability estimates through evaluation of V_1 along the solutions of (40)–(42), providing several steps (even though some of which may be similar to the ones from the proof of Theorem 2.6 in [19]), which are detailed within Eqs. (43)–(51).

Proof of Theorem 1. Employing Lemma 1 one can conclude from (7) that bound (37) holds with $\mu = \mu\left(t_1^*\right)$. Applying Lemma 2, where μ is updated according to (13) (and thus, it is fixed on any interval of the form $\left(t_1^*+(i-1)T, t_1^*+iT\right]$, $i=1,2,\ldots$), one can conclude that at time instant t_1^*+T relation (39) holds with $\mu = \mu\left(t_1^*\right)$. One could then apply again Lemma 2 for $t_1^*+T < t \leq t_1^*+2T$ since from (13) it follows that $\mu(t) = \Omega\mu\left(t_1^*\right)$, $t_1^*+T < t \leq t_1^*+2T$. In fact, the above procedure shows that the following holds

$$\sup_{0\leq x\leq D}\left|\mathrm{e}^{(x-D)\frac{\nu\ln(1+\lambda)}{D}} w\left(x, t_1^*+iT\right)\right| \leq \frac{1}{(1+\lambda)^{\nu}} M_2 M\Omega^{i}\mu\left(t_1^*\right), \quad i=1,2,\ldots. \quad (55)$$

From estimate (38) in Lemma 2 it then follows for $t_1^*+(i-1)T < t \leq t_1^*+iT$, $i=1,2,\ldots$, that

$$\sup_{0\leq x\leq D}\left|\mathrm{e}^{(x-D)\frac{\nu\ln(1+\lambda)}{D}} w(x,t)\right| \leq \max\left\{\sup_{0\leq x\leq D}\left|\mathrm{e}^{(x-D)\frac{\nu\ln(1+\lambda)}{D}} \times w\left(x, t_1^*+(i-1)T\right)\right|, \frac{\Omega M_2 M\mu(t)}{(1+\lambda)^{\nu}}\right\}. \quad (56)$$

Using (55) and (13) (which implies that $\mu(t) = \Omega^{i-1}\mu\left(t_1^*\right)$, $t_1^*+(i-1)T < t \leq t_1^*+iT$) we arrive at

$$\sup_{0\leq x\leq D}\left|\mathrm{e}^{(x-D)\frac{\nu\ln(1+\lambda)}{D}} w(x,t)\right| \leq \frac{1}{(1+\lambda)^{\nu}} M_2 M\Omega^{i-1}\mu\left(t_1^*\right), \quad t_1^*+(i-1)T < t \leq t_1^*+iT, \quad (57)$$

for $i=1,2,\ldots$, where we used the fact that $\Omega < 1$. Therefore,

$$\|w(t)\|_\infty \leq \frac{M_2 M}{\Omega}\Omega^{\frac{t-t_1^*}{T}}\mu\left(t_1^*\right), \quad t > t_1^*, \quad (58)$$

which in turn implies that

$$\|w(t)\|_\infty \leq \frac{M_2 M}{\Omega}\mu\left(t_1^*\right)\mathrm{e}^{\frac{\ln\Omega}{T}\left(t-t_1^*\right)}, \quad t > t_1^*. \quad (59)$$

Furthermore, using (13) we have that $\mu\left(t_1^*\right) \leq \max\left\{1, D\bar{M}_1\right\} \mathrm{e}^{2\bar{M}_1\tau}\mathrm{e}^{2\bar{M}_1 t_1^*}\mu_0$, and hence,

$$\|w(t)\|_\infty \leq \mu_0\max\left\{1, D\bar{M}_1\right\}\frac{M_2 M}{\Omega}\mathrm{e}^{2\bar{M}_1\tau}\mathrm{e}^{\left(2\bar{M}_1-\frac{\ln\Omega}{T}\right)t_1^*}\mathrm{e}^{\frac{\ln\Omega}{T}t}, \quad t > t_1^*. \quad (60)$$

Estimate (31) and relation (7) imply that

$$\|w(t)\|_\infty \leq \max\left\{1, D\bar{M}_1\right\} M_1\mathrm{e}^{\bar{M}_1 t_1^*}\|u(0)\|_\infty, \quad 0\leq t\leq t_1^*. \quad (61)$$

Combining estimates (60), (61), we arrive with the help of (7) at

$$\|u(t)\|_\infty \leq \max\left\{\mathrm{e}^{\bar{M}_1 t_1^*}, \|u(0)\|_\infty\right\}\bar{M}_2\mathrm{e}^{\bar{M}_1 t_1^*}\mathrm{e}^{-\frac{\ln\Omega}{T}t_1^*}\mathrm{e}^{\frac{\ln\Omega}{T}t}, \quad t\geq 0, \quad (62)$$

where $\bar{M}_2 = \frac{\max\{1, D\bar{M}_1\}}{M_2}\max\left\{\frac{M_2 M}{\Omega}\mathrm{e}^{2\bar{M}_1\tau}\mu_0, M_1\right\}$. Since from Lemma 1 $t_1^* \leq \frac{1}{\bar{M}_1}\ln\frac{\frac{M_1}{\mu_0}\|u(0)\|_\infty}{M_2\left(\frac{M}{(1+\lambda)^{\nu}}\frac{M_2}{M_1}-2\Delta\right)}$ and as $-\frac{\ln\Omega}{T} > 0$ (since $0 < \Omega < 1$) we get using the properties of the logarithmic function and defining $\bar{M}_3 = \frac{\frac{M_1}{\mu_0}}{M_2\left(\frac{M}{(1+\lambda)^{\nu}}\frac{M_2}{M_1}-2\Delta\right)}$ that

$$\mathrm{e}^{\bar{M}_1 t_1^*} \leq \bar{M}_3\|u(0)\|_\infty \quad (63)$$

$$\mathrm{e}^{-\frac{\ln\Omega}{T}t_1^*} \leq \mathrm{e}^{\ln\left(\left(\bar{M}_3\|u(0)\|_\infty\right)^{-\frac{\ln\Omega}{T}\frac{1}{\bar{M}_1}}\right)} \leq \bar{M}_3^{-\frac{\ln\Omega}{T}\frac{1}{\bar{M}_1}}\|u(0)\|_\infty^{-\frac{\ln\Omega}{T}\frac{1}{\bar{M}_1}}. \quad (64)$$

Therefore, using (63), (64) we obtain

$$\max\left\{\mathrm{e}^{\bar{M}_1 t_1^*}, \|u(0)\|_\infty\right\} \leq \max\left\{\bar{M}_3, 1\right\}\|u(0)\|_\infty \quad (65)$$

$$\mathrm{e}^{\bar{M}_1 t_1^*}\mathrm{e}^{-\frac{\ln\Omega}{T}t_1^*} \leq \bar{M}_3^{1-\frac{\ln\Omega}{T}\frac{1}{\bar{M}_1}}\|u(0)\|_\infty^{1-\frac{\ln\Omega}{T}\frac{1}{\bar{M}_1}}. \quad (66)$$

Since from (18) it follows that $\frac{\ln\Omega}{T} = -\frac{\nu\ln(1+\lambda)}{D}$, from (62) we arrive, combining (65), (66) with the definitions of $\bar{M}_2$, $\bar{M}_3$, at (19).

Existence and uniqueness of solutions are shown capitalizing on the relation of first-order hyperbolic PDEs to integral delay equations [19,23]. Thus, utilizing the invertibility of the backstepping transformation and the regularity properties of the kernels, it suffices to show existence and uniqueness in variable η. In particular, for $0 \leq t \leq t_1^*$ Eq. (22) satisfies all assumptions of Theorem 2.1 in [19], and hence, employing estimate (30) there exists a unique solution $\eta \in L^\infty\left(\left[-D, t_1^*\right]; \mathbb{R}\right)$. For $t > t_1^*$, in each of the intervals $\left(t_1^* + (i-1)T, t_1^* + iT\right]$, $i = 1, 2, \ldots$, where μ is constant, the transformed closed-loop system is equivalent to (40)–(42). Therefore, as the initial condition $\bar{\eta}_{t_1^*} = \eta\left(t_1^* + s\right)$, $-D \leq s \leq 0$, satisfies $\bar{\eta}_{t_1^*} \in L^\infty([-D, 0]; \mathbb{R})$, one can employ Theorem 2.1 in [19] in each of the intervals $\left(t_1^* + (i-1)T, t_1^* + iT\right]$, $i = 1, 2, \ldots$ (since for fixed and bounded $\mu > 0$, the mapping f has the particular form (41) with q being locally Lipschitz and satisfying $q(0) = 0$), which, in combination with the stability estimate (19), guarantee existence and uniqueness of $\eta \in L^\infty\left(\left(t_1^*, +\infty\right); \mathbb{R}\right)$. In fact, those properties of q and the forms (22), (41) of f guarantee that η is continuous on each interval of the form $\left(t_1^* + (i-1)T, t_1^* + iT\right)$, $i = 1, 2, \ldots$, as well as on $\left(0, t_1^*\right)$ (and left continuous at $t_1^* + (i-1)T$, $i = 1, 2, \ldots$). □

4. Extension to input quantization

In the case where the control input is subject to quantization, while measurements of the PDE state are available, the hybrid feedback law is modified to

$$U(t) = \begin{cases} 0, & 0 \leq t \leq \bar{t}_1^* \\ \mu(t) q\left(\frac{1}{\mu(t)} \int_0^D k(D, x) u(x, t) dx\right), & t > \bar{t}_1^* \end{cases}, \tag{67}$$

where the tuning strategy for μ is given as

$$\mu(t) = \begin{cases} \max\left\{1, D\bar{M}_1\right\} e^{2\bar{M}_1 j\tau} \mu_0, & (j-1)\tau \leq t \leq j\tau, \quad 1 \leq j \leq \lfloor \frac{\bar{t}_1^*}{\tau} \rfloor + 1 \\ \mu\left(\bar{t}_1^*\right), & \bar{t}_1^* < t \leq \bar{t}_1^* + T \\ \Omega \mu\left(\bar{t}_1^* + (i-1)T\right), & \bar{t}_1^* + (i-1)T < t \leq \bar{t}_1^* + iT, \quad i = 2, \ldots \end{cases}, \tag{68}$$

and $\bar{t}_1^*$ is the first time instant at which the following holds

$$\left|u\left(x, \bar{t}_1^*\right)\right| \leq \frac{M}{(1+\lambda)^\nu} \frac{M_2}{M_1 M_3} \mu\left(\bar{t}_1^*\right), \quad \text{for all } x \in [0, D]. \tag{69}$$

Event (69) can be detected using the available measurements of the PDE state. Alternatively, one could verify whether relation $\frac{\|u(\bar{t}_1^*)\|_\infty}{\mu(\bar{t}_1^*)} \leq \frac{M}{(1+\lambda)^\nu} \frac{M_2}{M_1 M_3}$ is satisfied, which implies that (69) holds. The form of the tuning strategy for μ is identical to the case of measurements quantization. The difference lies in the fact that the switching instant $\bar{t}_1^*$ is now characterized by the detection of event (69).

Theorem 2. *Consider the closed-loop system consisting of the plant (1), (2) and the hybrid feedback law (67), (68) with parameters (15)–(18). If Δ and M satisfy $\frac{\Delta}{M} < \frac{M_2}{M_3(1+\lambda)^{\nu+1}}$, then for all $u(\cdot, 0) \in L^\infty([0, D]; \mathbb{R})$ there exists a unique solution $u(\cdot, t) \in L^\infty([0, D]; \mathbb{R})$, which satisfies*

$$\|u(t)\|_\infty \leq \bar{\gamma} \|u(0)\|_\infty^{2 + \frac{\nu \ln(1+\lambda)}{D\bar{M}_1}} e^{-\frac{\nu \ln(1+\lambda)}{D} t}, \quad t \geq 0, \tag{70}$$

where $\bar{\gamma} = \frac{\max\{1, D\bar{M}_1\}}{M_2} \max\left\{\frac{M_2 M}{\Omega M_3} e^{2\bar{M}_1 \tau} \mu_0, M_1\right\} \max\left\{\frac{M_1^2 (1+\lambda)^\nu M_3}{\mu_0 M M_2^2}, 1\right\} \left(\frac{M_1^2 (1+\lambda)^\nu M_3}{\mu_0 M M_2^2}\right)^{1 + \frac{\nu \ln(1+\lambda)}{D\bar{M}_1}}$.

For specified data of the plant, i.e., for specified parameters M_1, M_2, and M_3, the condition of Theorem 1 is more restrictive, as compared with the respective condition in Theorem 2, in terms of the allowable range of the quantizer parameters (expressed as the allowable range of quantity $\frac{\Delta}{M}$), since $M_3 < M_1$ and $(1+\lambda)^{\nu+1} \leq (1+\lambda)^\nu \max\{1+\lambda, 2\}$ (for $0 < \nu < 1$ and $\lambda > 0$). In both cases, the degrees of freedom for satisfying the respective conditions are identical and, specifically, dependent on the choice of parameters λ and ν.

The proof of Theorem 2 is based on the following two lemmas, whose proofs employ similar arguments to the case of measurements quantization.

Lemma 3. *There exists a time instant $\bar{t}_1^*$, satisfying $\bar{t}_1^* \leq \frac{1}{\bar{M}_1} \ln \frac{\frac{M_1}{\mu_0} \|u(0)\|_\infty}{\frac{M}{(1+\lambda)^\nu} \frac{M_2^2}{M_1 M_3}}$, such that relation (69) holds.*

Proof. Employing the backstepping transformation (5) and its inverse (6) together with the feedback law (67), for $0 \leq t \leq \bar{t}_1^*$ the transformed closed-loop system is identical to the transformed closed-loop system to the case of state quantization, defined in (3), (4), with $U \equiv 0$, and hence, proceeding exactly as in the proof of Lemma 1 we obtain

$$\|w(t)\|_\infty \leq \max\left\{1, D\bar{M}_1\right\} e^{\bar{M}_1 t} \|w(0)\|_\infty, \quad 0 \leq t \leq \bar{t}_1^*. \tag{71}$$

Choosing the switching signal μ according to (68) one can conclude that there exists a time instant $\bar{t}_1^*$ such that

$$\frac{\left\|w\left(\bar{t}_1^*\right)\right\|_\infty}{\mu\left(\bar{t}_1^*\right)} \leq \frac{M}{(1+\lambda)^\nu} \frac{M_2^2}{M_1 M_3}, \tag{72}$$

and hence, since from (7) for all $0 \leq x \leq D$ it holds that $\frac{|u(x, \bar{t}_1^*)|}{\mu(\bar{t}_1^*)} \leq \frac{\|u(\bar{t}_1^*)\|_\infty}{\mu(\bar{t}_1^*)} \leq \frac{1}{M_2} \frac{\|w(\bar{t}_1^*)\|_\infty}{\mu(\bar{t}_1^*)}$, we obtain that relation (69) holds. □

Lemma 4. *Under the constraint for Δ and M stated in Theorem 2, the solutions of the transformed closed-loop system (3), (4), (67), for fixed $\mu > 0$, which satisfy*

$$\sup_{0 \leq x \leq D} \left|e^{(x-D)\frac{\nu \ln(1+\lambda)}{D}} w\left(x, \bar{t}_1^*\right)\right| \leq \frac{1}{(1+\lambda)^\nu M_3} M_2 M \mu, \tag{73}$$

they also satisfy for $\bar{t}_1^* < t < \bar{t}_1^* + T$

$$\sup_{0 \leq x \leq D} \left|e^{(x-D)\frac{\nu \ln(1+\lambda)}{D}} w(x, t)\right| \leq \max\left\{\sup_{0 \leq x \leq D} \left|e^{(x-D)\frac{\nu \ln(1+\lambda)}{D}} w\left(x, \bar{t}_1^*\right)\right|, \frac{\Omega M_2 M \mu}{(1+\lambda)^\nu M_3}\right\}. \tag{74}$$

Moreover, the following holds

$$\sup_{0 \leq x \leq D} \left|e^{(x-D)\frac{\nu \ln(1+\lambda)}{D}} w\left(x, \bar{t}_1^* + T\right)\right| \leq \Omega M_2 M \mu \frac{1}{(1+\lambda)^\nu M_3}. \tag{75}$$

Proof. Under the feedback law (67) for $\bar{t}_1^* < t \leq \bar{t}_1^* + T$ and fixed μ we get the following integral delay equation

$$\eta(t) = \bar{f}\left(\eta_t, \mu\right), \tag{76}$$

where

$$\bar{f}\left(\eta_t, \mu\right) = \mu\left(q\left(\frac{\int_{-D}^0 k(D, s+D) G(t, s) ds}{\mu}\right) - \frac{1}{\mu} \int_{-D}^0 k(D, s+D) G(t, s) ds\right), \tag{77}$$

and G is defined in (42), which is equivalent to the transformed system (3), (4), under the control law (67) (where u is expressed in terms of w through (6)). As long as $\frac{\Omega}{(1+\lambda)^\nu} \frac{M_2}{M_3} M\mu \leq \|\eta\| \leq \frac{M_2}{M_3} M\mu$, using the property P1 in Section 2.2 of the quantizer and

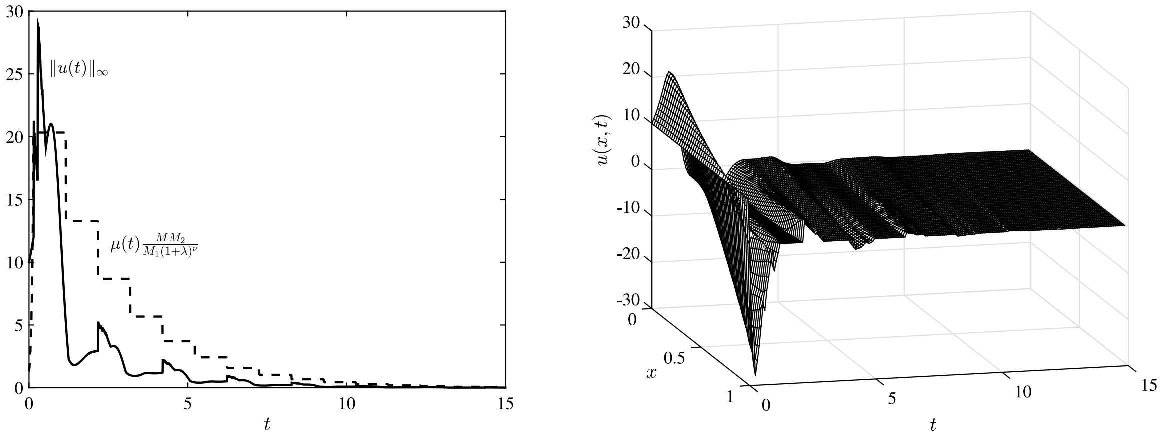

Fig. 2. Left: The norm $\|u(t)\|_\infty$ (solid line) of the closed-loop system (1), (2), with parameters $D=1$, $\bar{g}\equiv 0$, and $g(x)=g=1.25$, for all $x\in[0,1]$, under the feedback law (12), (13), (15)–(18), (81), with parameters $M=2$, $\Delta=\frac{M}{40}$, $\lambda=\nu=\frac{3}{4}$, $M_1=1+ge^g$, $M_2=\frac{1}{1+g}$, and $\mu_0=1$. The switching signal $\mu(t)\frac{MM_2}{M_1(1+\lambda)^\nu}$ (dashed line) is also shown, with $\Omega=0.65$ and $T=1.01$. Right: The corresponding state of the closed-loop system. The control effort is shown for $x=1$.

the left-hand side of bound (7) we arrive with (77) at

$$\left|\bar{f}(\eta,\mu)\right| \le \mu\Delta \le \frac{(1+\lambda)^\nu M_3\Delta}{\Omega M_2 M}\|\eta\|. \tag{78}$$

Proceeding as in the case of the corresponding part of the proof of Lemma 2 we arrive at (51), and hence, with the definition of V_1 we obtain from (73) that $\|\eta_t\|\le\frac{M_2}{M_3}M\mu$, $\bar{t}_1^*<t\le\bar{t}_1^*+T$. Moreover, using (51), from the definition of V_1 and (18) we get that

$$\sup_{-D\le s\le 0} e^{s\frac{\nu\ln(1+\lambda)}{D}}\left|\eta\left(\bar{t}_1^*+T+s\right)\right| \le \Omega \sup_{-D\le s\le 0} e^{s\frac{\nu\ln(1+\lambda)}{D}}\left|\eta\left(\bar{t}_1^*+s\right)\right|, \tag{79}$$

and hence, employing (73) we obtain (75), which also implies the following ultimate bound

$$\|\eta_{\bar{t}_1^*+T}\|\le\Omega\frac{M_2}{M_3}M\mu. \tag{80}$$

The rest of the proof employs identical arguments to the proof of Lemma 2. □

Proof of Theorem 2. The proof of Theorem 2 follows the same lines as the corresponding part of the proof of Theorem 1, employing Lemmas 3 and 4. □

5. Simulation results

We consider system (1), (2) with $D=1, f\equiv 0$, and $g(x)=g=1.25$, for all $x\in[0,1]$. Computing the eigenvalues of the generator associated with the open-loop system, it is shown that there is a real, positive eigenvalue $\sigma\approx 0.46$, satisfying $e^\sigma(\sigma-g)+g=0$. The quantizer is defined component-wise for each $x\in[0,1]$ as

$$q\left(\frac{u(x)}{\mu}\right)=\begin{cases} M, & \frac{u(x)}{\mu}>M \\ -M, & \frac{u(x)}{\mu}<-M \\ \Delta\left\lfloor\frac{u(x)}{\mu\Delta}+\frac{1}{2}\right\rfloor, & -M\le\frac{u(x)}{\mu}\le M \end{cases}, \tag{81}$$

with $M=2$ and $\Delta=\frac{M}{40}$. The switching signal μ is updated according to (13) with $\lambda=\nu=\frac{3}{4}$. The initial condition is chosen, for simplicity, as constant, namely, $u(x,0)=10$, for all $x\in[0,1]$. At time $t_1^*=0.14$ "capture" is guaranteed, i.e., event $\left|q_\mu\left(u\left(x,t_1^*\right)\right)\right| \le\left(\frac{M}{(1+\lambda)^\nu}\frac{M_2}{M_1}-\Delta\right)\mu\left(t_1^*\right)$, for all $x\in[0,1]$, is detected, through verifying that relation $\left\|q_\mu\left(u\left(t_1^*\right)\right)\right\|_\infty\le\left(\frac{M}{(1+\lambda)^\nu}\frac{M_2}{M_1}-\Delta\right)\mu\left(t_1^*\right)$ holds.

We show in Fig. 2 the supremum norm of the state of the closed-loop system together with the switching signal $\mu(t)\frac{MM_2}{M_1(1+\lambda)^\nu}$, as well as the response of the state, where, in particular, at $x=1$ we show the control input signal. The response of the closed-loop system is computed numerically employing a Lax–Friedrichs scheme (see, e.g., [26]) with time- and spatial-discretization steps equal to 0.005 and 0.02, respectively. The integral incorporated in the backstepping controller (12) is computed numerically using a left endpoint rule, while the backstepping kernels are given explicitly as $k(x,y)=-ge^{g(x-y)}$ and $l(x,y)=-g$.

In order to further illustrate the significance of the proposed control design methodology, we show in Figs. 3 and 4 the responses of the closed-loop systems in the case in which the nominal backstepping controller is employed, i.e., when the effect of state measurements quantization is left uncompensated. For a fixed, small value of μ, namely $\mu=0.5$, the state of the closed-loop system, under the nominal control law, grows unbounded, as shown in Fig. 3, since the initial condition is outside the range of the quantizer, which is equal to $M\mu$. For a fixed, large value of μ, namely $\mu=100$, the state of the closed-loop system, under the nominal control law, remains bounded, which may be also attributed to the input-to-state stability (with respect to a boundary disturbance) property of the nominal backstepping controller (see, e.g., Section 4 in [19]). Nevertheless, asymptotic stabilization is not possible, as shown in Fig. 4, since the quantization error is large, in particular, equal to $\Delta\mu$. In fact, it appears as though the system goes into a limit cycle, since, due to quantization effect, the control input vanishes when the state lies within a certain region around zero, which results in the state to grow (as the open-loop system is unstable), until the quantizer may switch to a non-zero value. Such a closed-loop system behavior is consistent both with results reported in the finite-dimensional case, see, e.g., [18,27], as well as with the results for hyperbolic systems in [15,16] (which do not explicitly aim at compensation of the quantization effect in order to achieving asymptotic stabilization).

Last but not least, in Fig. 5 we also show the corresponding closed-loop response in the case of input quantization, i.e., under the feedback law (67)–(69), where the quantizer q in this case is

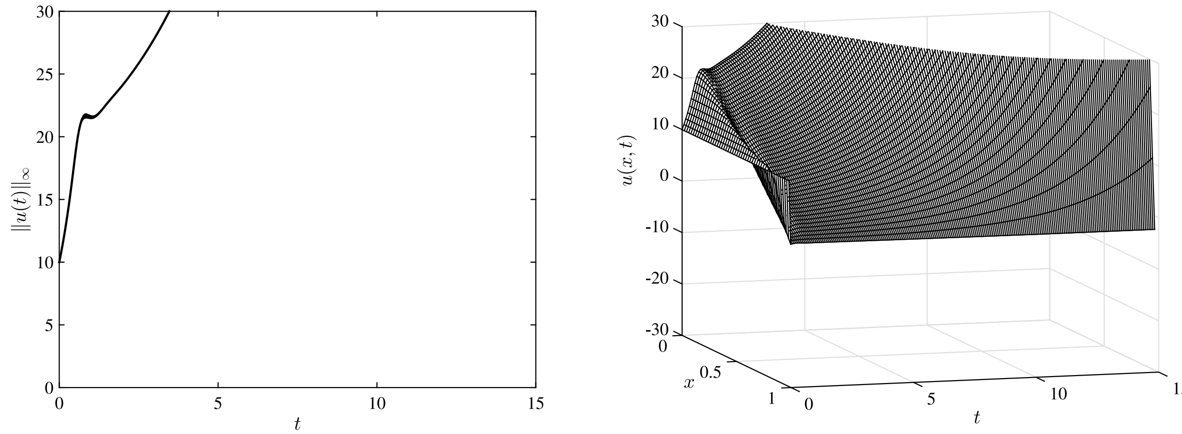

Fig. 3. Left: The norm $\|u(t)\|_\infty$ of the closed-loop system (1), (2), with parameters $D = 1$, $\bar{g} \equiv 0$, and $g(x) = g = 1.25$, for all $x \in [0, 1]$, under the nominal feedback law $U(t) = \mu \int_0^D k(D, x)\, q\left(\frac{u(x,t)}{\mu}\right) dx$, for fixed $\mu = 0.5$ and q defined in (81), with parameters $M = 2$, $\Delta = \frac{M}{40}$. Right: The respective state of the closed-loop system.

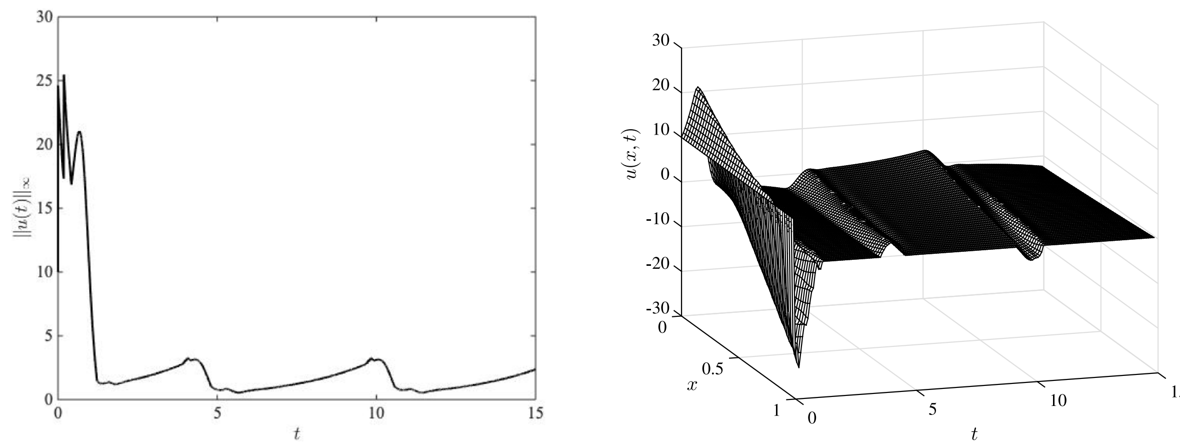

Fig. 4. Left: The norm $\|u(t)\|_\infty$ of the closed-loop system (1), (2), with parameters $D = 1$, $\bar{g} \equiv 0$, and $g(x) = g = 1.25$, for all $x \in [0, 1]$, under the nominal feedback law $U(t) = \mu \int_0^D k(D, x)\, q\left(\frac{u(x,t)}{\mu}\right) dx$, for fixed $\mu = 100$ and q defined in (81), with parameters $M = 2$, $\Delta = \frac{M}{40}$. Right: The respective state of the closed-loop system.

defined as

$$q\left(\frac{\bar{U}}{\mu}\right) = \begin{cases} M, & \frac{\bar{U}}{\mu} > M \\ -M, & \frac{\bar{U}}{\mu} < -M \\ \Delta \left\lfloor \frac{\bar{U}}{\mu\Delta} + \frac{1}{2} \right\rfloor, & -M \le \frac{\bar{U}}{\mu} \le M \end{cases}, \tag{82}$$

and, in particular, it has only one component (since it takes as argument the value of the scalar quantity $\frac{\bar{U}}{\mu} = \frac{\int_0^D k(D,x)u(x)dx}{\mu}$).

6. Future work

Although the update law for the zoom variable of the quantizer is based on time-dependent switching, a, potentially, more robust strategy would be to design a state-dependent switching rule (see, e.g., [28], for design of a state-dependent switching strategy in boundary controlled, hyperbolic PDE systems), on the basis of the available quantized measurements, for determining when the state enters a certain region. For a control strategy that would be based on state-dependent switching, one would determine the switching instants, say t_i^*, $i = 2, 3, \ldots$, at which the value of μ is updated, based on the satisfaction of an event-based criterion indicating when the state enters a desired region. In the case of state measurements quantization such events could be determined for $i = 1, 2, \ldots$ employing conditions of the form

$$\left|q_{\mu(t_i^*)}\left(u\left(x, t_{i+1}^*\right)\right)\right| \le \left(\frac{M}{(1+\lambda)^\nu}\frac{M_2}{M_1}\Omega - \Delta\right)\mu\left(t_i^*\right), \quad \text{for all } x \in [0, D], \tag{83}$$

(or, alternatively, utilizing conditions $\left\|q_{\mu(t_i^*)}\left(u\left(t_{i+1}^*\right)\right)\right\|_{L^\infty} \le \left(\frac{M}{(1+\lambda)^\nu}\frac{M_2}{M_1}\Omega - \Delta\right)\mu\left(t_i^*\right)$), which would also guarantee (using the arguments within the proof of Lemma 1 and relation (7)) that bound (39) holds (with μ replaced by $\mu\left(t_1^*\right)$ and $t_1^* + T$ replaced by t_2^*). Such an approach could be suitable for a continuously adjusted control input. Furthermore, employment of such a strategy, would potentially also require establishment of avoidance of Zeno behavior in the closed-loop system.

As the issue of existence and uniqueness of closed-loop solutions in the case in which the quantizer is only piecewise constant function, taking finitely many values, may, potentially, become more tractable when the control input is applied through a zero-order hold (see, e.g., [8]), which may also be the case in more realistic, practical scenarios, a potential next step would be to consider, simultaneously, the effect of sampling and quantization. In the case in which an event-triggered mechanism may be utilized, updating the values of control input at discrete time

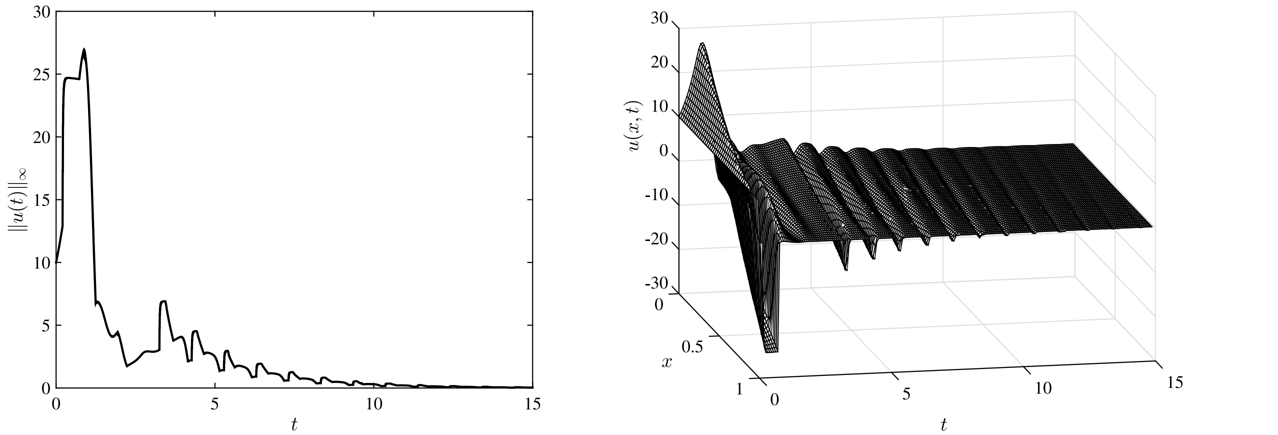

Fig. 5. Left: The norm $\|u(t)\|_\infty$ of the closed-loop system (1), (2), with parameters $D = 1$, $\bar{g} \equiv 0$, and $g(x) = g = 1.25$, for all $x \in [0, 1]$, under the feedback law (67), (68), (15)–(18), (82), with parameters $M = 2$, $\varDelta = \frac{M}{40}$, $\lambda = \nu = \frac{3}{4}$, $M_1 = 1 + ge^g$, $M_2 = \frac{1}{1+g}$, and $\mu_0 = 1$. Right: The corresponding state of the closed-loop system. The control effort is shown for $x = 1$.

instants, one may have to combine the event-based strategy for updating the values of μ (e.g., according to (83)), with an event-triggered strategy, potentially based on an approach as the ones presented in, e.g., [7,29], for updating the values of the control input U, possibly employing a different criterion for deriving the triggering instants.

CRediT authorship contribution statement

Nikolaos Bekiaris-Liberis: Conceptualization, Formal analysis, Investigation, Methodology, Software, Validation, Visualization, Writing - original draft, Writing - review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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